

LAB #1: TIME AND FREQUENCY RESPONSES OF SERIES *RLC* CIRCUITS

Objective:

To investigate the *step*, *impulse*, and *frequency responses* of series *RLC* circuits. To compare *experimental* results with *theory* and *LTspice simulations*, and to account for possible differences. To gain familiarity with *Bode plots*.

Components:

1 × 3.9-mH inductor, 1 × 10-nF capacitor, 1 × 1 kΩ potentiometer, and resistors: 1 × 10 Ω, 1 × 39 Ω, 1 × 1.0 kΩ, and 1 × 2.0 kΩ, 1 × 3.9 kΩ, (all 5%, ¼ W).

Instrumentation:

An *RLC* meter, a waveform generator (square-wave, pulse, and sine-wave), and a dual-trace oscilloscope.

References:

1. Franco, Sergio *Electric Circuits Fundamentals*, Oxford University Press, 1995 (Chs. 9 and 14).
2. Roberts, Gordon W., and Sedra, Adel S., *SPICE*, 2nd Ed.; Oxford University Press, 1997.

PART I – THEORETICAL BACKGROUND

RLC circuits are classical examples of *second-order systems*. Together with their mass-spring-dashpot mechanical analog, they are used to illustrate fundamental systems-theory concepts and techniques, such as Laplace-transform techniques and resonance.

The current response of the *series RLC circuit* of Fig. 1 is found via Laplace-transform techniques as $I(s) = Y(s)V(s)$, where $I(s)$ and $V(s)$ are the Laplace transforms of $i(t)$ and $v(t)$, s is the *complex frequency*, and $Y(s)$ is the *complex admittance*, that is, the reciprocal of the *complex impedance* $Z(s)$,

$$Y(s) = \frac{1}{Z(s)} = \frac{1}{R + sL + \frac{1}{sC}} = \frac{sC}{s^2LC + sRC + 1}$$

This function is usually expressed in the standardized form

$$Y(s) = \frac{1}{R} \times \frac{2\zeta(s/\omega_0)}{(s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1} \quad (1)$$

where

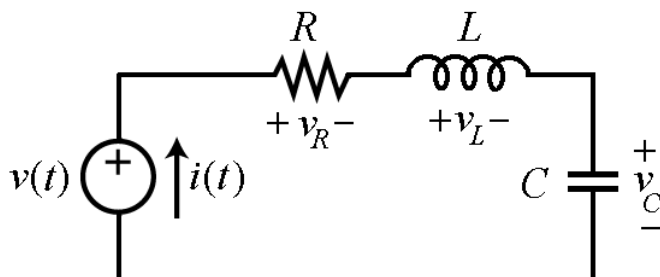


Fig.1 – Series *RLC* circuit.

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad \zeta = \frac{R}{2} \sqrt{\frac{C}{L}} \quad (2)$$

Here, ω_0 is the *undamped natural frequency*, in rad/s, and ζ is the *damping ratio*, dimensionless. The values of these parameters are set by those of the components making up the circuit.

The values of s for which the denominator of $Y(s)$ becomes zero are called the *poles* of $Y(s)$, and therefore, the *zeros* of $Z(s)$. They are easily found to be

$$s_{1,2} = \omega_0 \left(-\zeta \pm \sqrt{\zeta^2 - 1} \right) \quad (3)$$

We have the following significant cases:

- For $\zeta > 1$, the poles are *real and distinct*, and the system is said to be *overdamped*.
- For $\zeta = 1$, the poles are *real and coincident*, and the system is said to be *critically damped*.
- For $0 < \zeta < 1$, the poles are *complex conjugate*, or $s_{1,2} = \omega_0(-\zeta \pm j\sqrt{1-\zeta^2})$, where $j^2 = -1$. The system is now said to be *underdamped*.

In each of the above cases the poles lie in the *left-half* of the complex plane s . For $\zeta = 0$, the poles lie *right on the imaginary axis*, and the system is said to be *undamped*. It is apparent that varying R while keeping L and C constant will move the poles around in the complex plane.

Systems theory indicates that the response $i(t)$ to a given excitation $v(t)$ can be found as $i(t) = \mathbf{L}^{-1}\{I(s)\} = \mathbf{L}^{-1}\{Y(s)V(s)\}$, where \mathbf{L}^{-1} indicates *inverse Laplace transformation*. The responses of greatest practical interest in engineering are the *impulse*, the *step*, and the *ac* or *frequency responses*. The current response $i(t)$ is readily visualized with the oscilloscope by observing the voltage $v_R(t)$ across the resistance R ; then, $i(t) = v_R(t)/R$. Of great interest are also the capacitance and inductance responses $v_C(t)$ and $v_L(t)$.

The Transient Response:

Figure 3 shows the *step* or *transient* response across C for three different values of ζ . It can be proved that for $0 < \zeta < 1$, this response is a *damped sinusoid* with the frequency

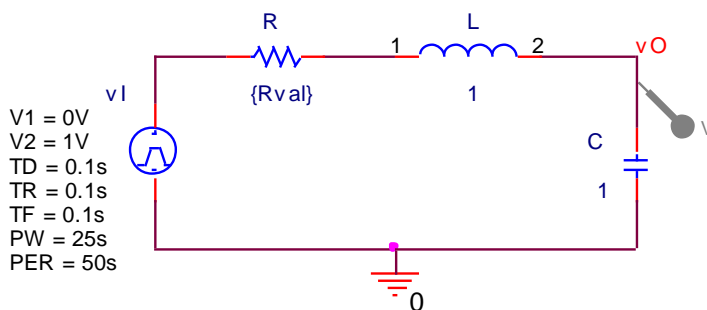


Fig. 2 – PSpice circuit to display the *step response* across C .

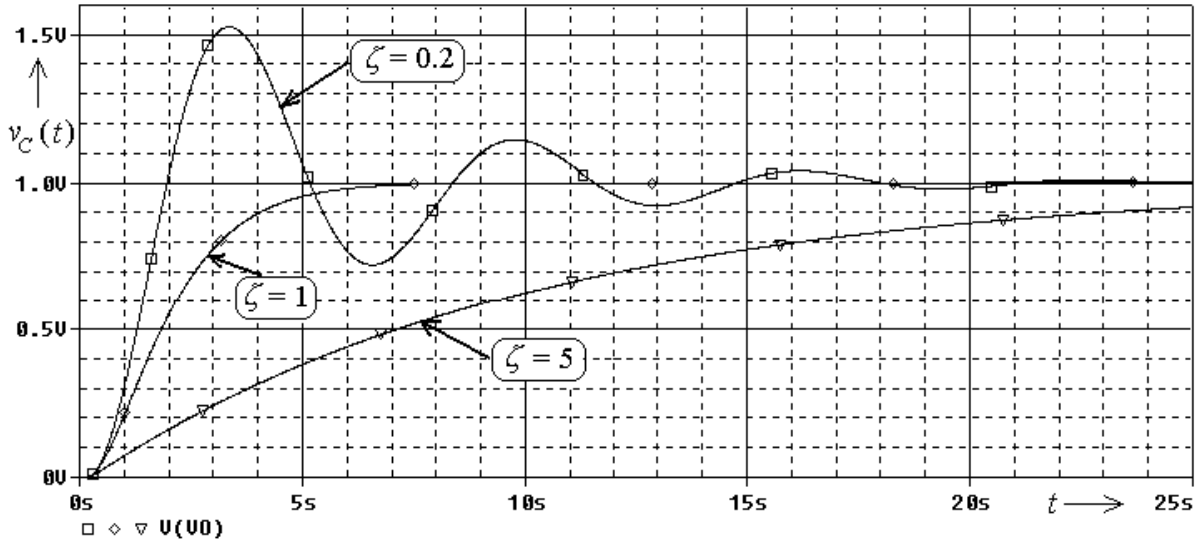


Fig. 3 – Step or transient response across C for different values of ζ .

$$\omega_d = \omega_0 \sqrt{1 - \zeta^2} \quad (4)$$

called the *damped frequency*. We also observe the presence of *overshoot*, defined as

$$\text{OS}(\%) = 100 \frac{v_{O(\text{peak})} - v_{O(\infty)}}{v_{O(\infty)}}$$

where $v_{O(\infty)}$ is the value of v_O in the limit $t \rightarrow \infty$. The overshoot is related to ζ as

$$\text{OS}(\%) = 100e^{-\pi\zeta / \sqrt{1-\zeta^2}} \quad (5)$$

The *smaller* the value of ζ , the *higher* the overshoot and the *longer* it takes for the oscillation to die out. In the limit $\zeta \rightarrow 0$ we have a *sustained oscillation* with undamped natural frequency ω_0 . If ζ is gradually increased from zero, the oscillation will die out more and more rapidly until the point is reached where there will be no more oscillation. This point corresponds to *critical damping*, or $\zeta = 1$. For $\zeta > 1$, not only is there no oscillation, but the system takes even a longer time to reach its steady state.

Frequency Response:

Systems theory indicates that the **frequency response** of a circuit is found by letting $s \rightarrow j\omega$ in its transfer function. In this case it is also more common to work with the parameter $Q = 1/(2\zeta)$, after which our expression above becomes

$$Y(j\omega) = \frac{1}{R} \times H_{\text{BP}}(j\omega)$$

where

$$H_{\text{BP}}(j\omega) = \frac{j(\omega/\omega_0)}{1 - (\omega/\omega_0)^2 + j(\omega/\omega_0)/Q} \quad (6)$$

and

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad Q = \frac{1}{R} \sqrt{\frac{L}{C}} \quad (7)$$

The function $H_{BP}(j\omega)$ is called the *standard second-order band-pass function*.

To investigate the frequency response of our circuit, we apply an ac voltage of the type

$$v_i(t) = V_{im} \cos \omega t \quad (8a)$$

and we observe the response $v_o(t) = Ri(t)$ across the resistor, which is an ac voltage of the type

$$v_o(t) = V_{om} \cos(\omega t + \phi) \quad (8b)$$

Here, V_{im} and V_{om} are the *peak amplitudes* (in V), ω is the *angular frequency* (in rad/s), and ϕ is the *phase angle* (in degrees). The parameters of the response are related to those of the applied voltage as

$$V_{om} = |H_{BP}| \times V_{im} \quad \phi = \angle H_{BP} \quad (9)$$

where $|H_{BP}|$ and $\angle H_{BP}$ are, respectively, the *magnitude* and *phase* of H_{BP} .

The LTspice circuit of Fig. 4 is used to visualize the *frequency response* across R for the case $\omega = 1$ rad/s. Again, you can simulate this circuit on your own by downloading its files from the Web, as mentioned earlier. Figure 5 shows the logarithmic plots of *magnitude* and *phase*, also called *Bode plots*, for three different values of Q . Each magnitude curve peaks at 0 dB for $\omega = \omega_0$, this being the reason why ω_0 is also called the *resonance frequency*. Moreover, each curve drops to -3 dB at two frequencies ω_L and ω_H such that

$$\omega_L = \omega_0 \left(\sqrt{1 + \frac{1}{4Q^2}} - \frac{1}{2Q} \right) \quad \omega_H = \omega_0 \left(\sqrt{1 + \frac{1}{4Q^2}} + \frac{1}{2Q} \right) \quad (10)$$

It is readily seen that these frequencies satisfy the condition $\omega_L \times \omega_H = \omega_0^2$, and that phase is $\pm 45^\circ$ at these frequencies. Moreover, the *half-power bandwidth*, defined as $BW = \omega_H - \omega_L$, is such that

$$Q = \frac{\omega_0}{BW} \quad (11)$$

It is apparent that the narrower the BW for a given ω_0 , the higher the value of Q . Consequently, Q provides a measure of the degree of *selectivity* of a filter such as ours.

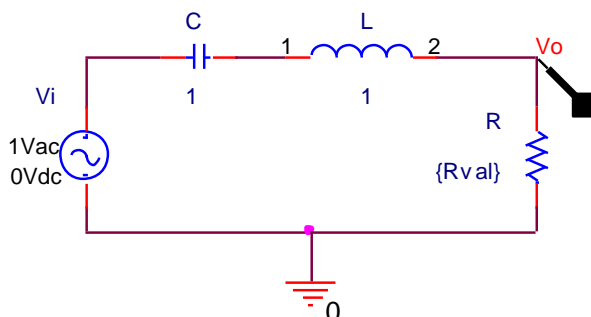


Fig.4 – PSpice circuit to display the *frequency response* across R .

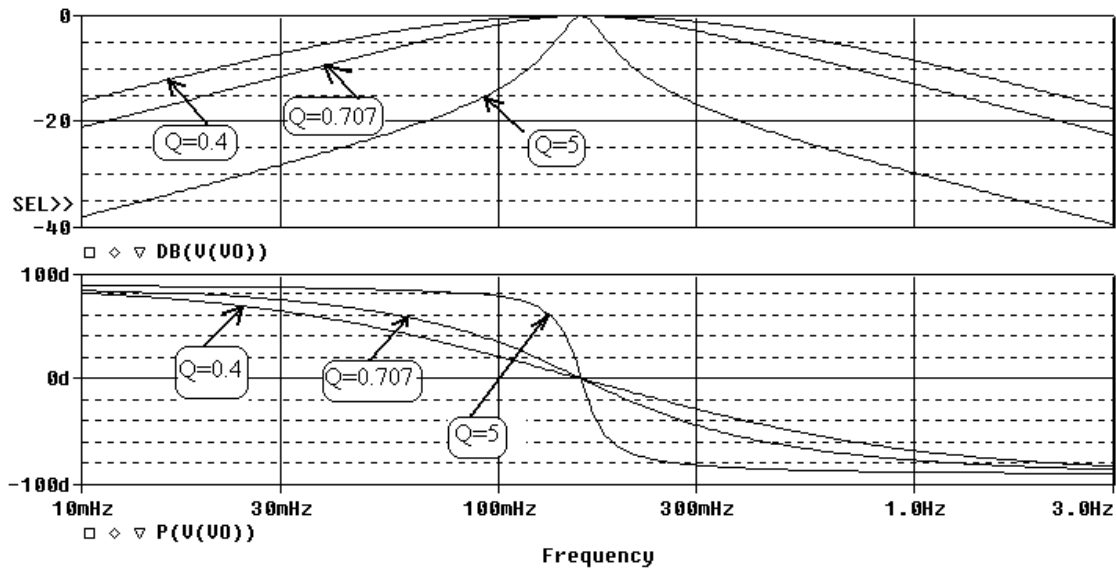


Fig. 5 – Bode plots for different values of Q

PART II – EXPERIMENTAL PART

To lower the output resistance R_s of the function generator (usually an ill-defined parameter on the order of $50\ \Omega$) to a *smaller and more predictable* value, we interpose a *voltage-divider adaptor* as shown in Fig. 6. Note that because of the internal ground connection of the oscilloscope, we must arrange the elements so that the element across which we wish to observe the response is *always* located at the site denoted as X_3 .

By Thevenin's theorem, the circuit reduces to the equivalent of Fig. 7 for the case in which we observe the response across the capacitor. Here, v_{OC} and R_{eq} are the parameters of the equivalent source, R_L is the winding resistance of the coil, and R_p is a variable resistance that we adjust to achieve specific values of ζ (or Q) for our circuit. This variable resistance is implemented via a potentiometer with the wiper connected to either one of its remaining terminals. If you need a value of R_p greater than the potentiometer's rating, use a suitable resistance in series. The expressions for ζ and Q derived above still

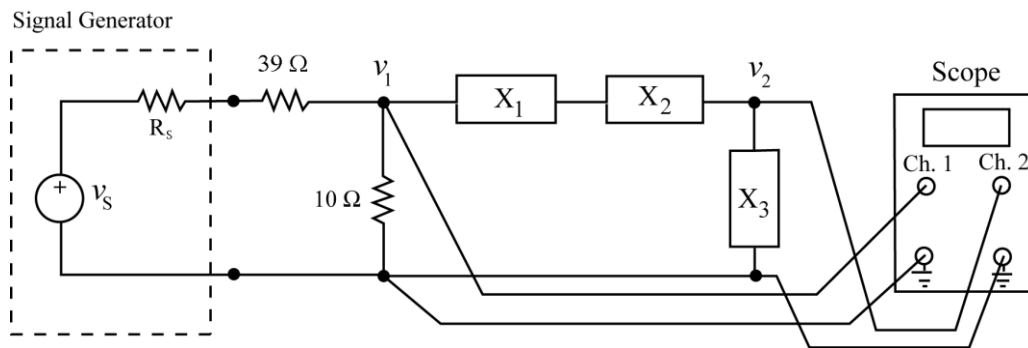


Fig. 6 – Experimental setup.

hold, provided we use

$$R = R_{eq} + R_L + R_p \quad (11)$$

In our case, $R_{eq} \cong (50 + 39)/10 \cong 9 \Omega$, R_L is measured, and is R_p is adjusted to specific values found via calculation, as we shall see shortly.

Initial Measurements and Calculations:

Henceforth, steps shall be identified by letters as follows: **C** for calculations, **M** for measurements, **S** for SPICE simulation.

M1: Using an *RLC* meter from the stockroom, measure and record the values of C (10nF), L (3.9mH), and R_L (the resistance of the coil). By how much do L and C differ from their nominal values?

C2: Calculate the undamped natural frequency $f_0 = 1/(2\pi\sqrt{LC})$. Please explain what the undamped natural frequency is.

Step Response:

C3: Calculate the three values of R using equation (2), and hence of $R_p = R - R_{eq} - R_L$, that result in $\zeta = 5$, $\zeta = 1$, and $\zeta = 0.2$, with the values of L and C measured in Step M1. For the case $\zeta = 0.2$, compute also the damped frequency f_d and overshoot OS(%) via Eqs. (4) and (5). Please explain what damped frequency and overshoot is. In other words, explain Eqs (4) and (5).

C4: Calculate the values of the **poles** for the three specified values of ζ , and show their complex-plane locations. Be neat and precise by keeping the Real axis with unit of (ω_0). For the complex-plane locations, please draw it in MATLAB or equivalent, or paste hand-written image of your work.

S5: Using LTspice, along with the component values of Steps M1 and C3, plot the response of the circuit of Fig. 7 to a 1-V step for the three specified values of ζ . For the case $\zeta = 0.2$, use the cursor facility of LTspice to estimate the overshoot OS(%) as well as the period T_d of the decaying oscillation and, hence, the damped frequency $f_d = 1/T_d$. Compare with the predicted values in Step C3. Are they close?

M6: Assemble the circuit of Fig. 6, with the coil as X_1 , the potentiometer as X_2 , and the capacitor as X_3 ,

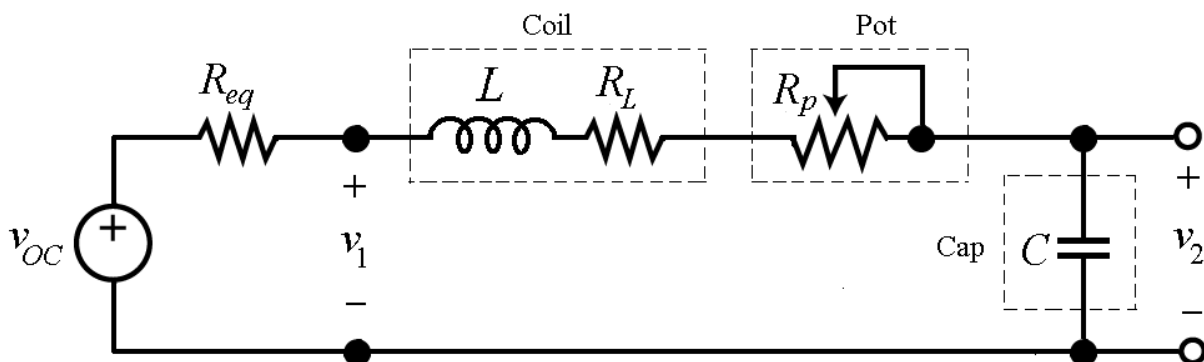


Fig. 7 – Equivalent circuit of Fig. 6 for the case in which X_3 is the capacitor ($R_{eq} = 9 \Omega$)

so that its equivalent is as in Fig. 7. Keeping in mind that $R_{eq} = 9 \Omega$, adjust R_p for $\zeta = 1$. Then, while monitoring v_1 with Ch.1 of the oscilloscope set on DC, adjust the waveform generator so that v_1 is a

square wave alternating between 0 V and 1 V with a frequency of about $f_0/10$, where f_0 is the undamped frequency of Step C2 (make sure you know where the 0-V baseline is on the screen!).

Now, observe and record the circuit's response by monitoring v_2 with Ch. 2 of the oscilloscope set on DC. Finally, compare with the response predicted via LTspice in Step S5, and account for any differences.

Note: In this and the subsequent steps, if the pot is not sufficient to achieve the desired resistance value, use suitable combinations of resistances in series with the pot. For instance, connecting the 1-k Ω pot in series with a 3 k Ω ordinary resistor will allow you to span the range of 3 k Ω to 4 k Ω .

M7: Repeat step M6, but with R_p adjusted for $\zeta = 5$. Provide a physical justification for why the response is now so sluggish. Compare the measurement with the simulation results.

M8: Repeat step M6, but with R_p adjusted for $\zeta = 0.2$. Also, from the oscilloscope trace, estimate f_d and OS(%) in a manner similar to Step S5, compare with those predicted in Step C3, and comment. Finally, provide a physical justification for why the response is now oscillatory. Compare the measurement with the simulation results.

Impulse Response:

Leaving the potentiometer setting as in Step M8 ($\zeta = 0.2$), interchange R_p and C so that the circuit becomes as in Fig. 8. Then, change the waveform generator settings so that v_1 is now a *pulse train* consisting of pulses each alternating between 0 V and 1 V with a pulse-width of about $0.1/f_0$, where f_0 is the undamped frequency calculated in Step C2. A pulse this narrow will provide a good approximation to the *impulse function* for our circuit. Moreover, to be able to see a repetitive trace on the oscilloscope, adjust the waveform generator settings so that the above pulses repeat with a frequency of about $10/f_0$.

M9: While triggering the oscilloscope from v_1 , observe v_2 with the other channel and record it (for best visualization, you may need to adjust the repetition frequency from the initial suggested value of $10/f_0$.) Next, measure the period T_d of the damped oscillation, calculate $1/T_d$, and compare with f_d of Step C3. Finally, justify the waveform for the response v_2 using physical insight.

Frequency Response:

To investigate this type of response we still use the circuit of Fig. 8, except that we change the waveform generator settings so that v_1 is now a *sinusoidal signal* with a *constant* peak amplitude of 1 V, 0-V DC, and variable frequency f .

You can measure f by (a) reading the frequency setting on the waveform generator, or (b) by measuring the period T with the oscilloscope and then computing $f = 1/T$, or (c) by using a frequency meter from the stockroom. It is up to each group to decide which method to pursue, and to justify your choice in the final report.

C10: Find the value of R , and, hence, of R_p , that results in $Q = 5$ (use Eq. 7) in the circuit of Fig. 8. Then, using Eqs. (10) and (11), calculate f_L , f_H , and the bandwidth BW . Please explain what bandwidth BW is.

S11: Using LTspice, along with the component values of Steps M1 and C10, generate the Bode Plots of the circuit of Fig. 8. Then, using the cursor facility of LTspice, estimate f_L and f_H , first as the -3-dB frequencies on the magnitude plot, then as the $\pm 45^\circ$ frequencies on the phase plot. Compare the resulting values of f_L , f_H , and BW against those of Step C10, and account for possible differences.

M12: While monitoring v_2 with the oscilloscope, vary the waveform generator's frequency f until v_2

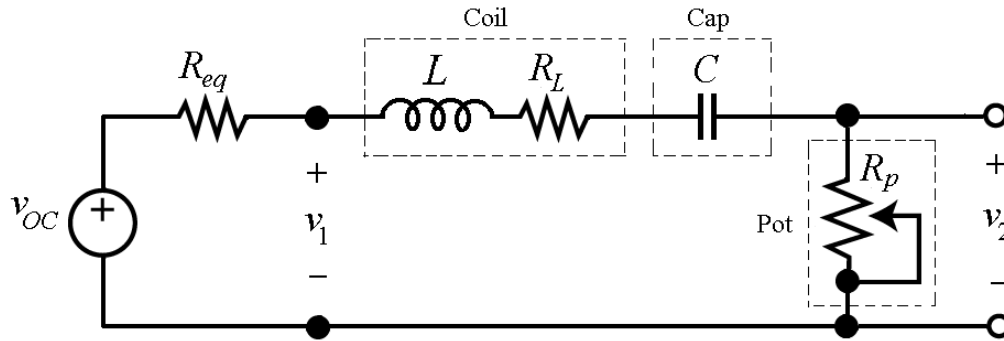


Fig. 8 – Circuit to investigate the *impulse* and *frequency* responses across R_p .

reaches its *maximum*, and record the value of f . This is the experimental value of f_0 . Compare with the calculated value of Step C2. Do they agree within their respective uncertainties? Account for possible differences! Also, how does the maximum amplitude of v_2 compare with the amplitude of v_1 ? Justify via suitable voltage-divider calculations!

M13: Vary the waveform generator's frequency f until the amplitude of v_2 is down to 70.7% of its maximum as found in Step M12. There are two such frequencies, namely, f_L and f_H . How do they compare with the calculated values of Step C10. Do they agree within their respective uncertainties?

M14: Repeat Step M13, except that now we shall find f_L and f_H as the $\pm 45^\circ$ frequencies. For phase measurements, use Channel 1 and Channel 2 of the oscilloscope for input and output. Which of the methods of estimating f_L and f_H do you think is the *most* and which the *least* dependable, and why? Please explain the method you used to find the phase measurements.

M15: Verify experimentally the following important properties:

- For $f \ll f_0$, *increasing* f by a factor of 10 *increases* amplitude also by a factor of 10, this being the reason why it is said that the slope of the magnitude curve is +20 dB/dec there.
- For $f \gg f_0$, *increasing* f by a factor of 10 *decreases* amplitude also by a factor of 10, this being the reason why the slope of the magnitude curve is said to be -20 dB/dec there.

M16: Interchange R_p and C so that we are back to the circuit of Fig. 7, to observe the response across C . Find the value of R that results in $Q = 1/\sqrt{2} = 0.707$, and hence adjust R_p accordingly. Then, by suitably varying the waveform generator's frequency f while leaving amplitude and DC offset unchanged, find experimentally the following:

- The -3-dB frequency f_{-3dB}
- The low-frequency amplitude of v_2
- The amplitude of v_2 at $10f_{-3dB}$ and $100f_{-3dB}$

S17: Using LTspice, plot the magnitude response across C . Then, compare with the values in M16.