Lecture 4 : Reconstruction with erasure.

When we received the data $\{\langle x, \varphi_i \rangle\}_{i=1}^{M}$ without any loss, then

$$x = \sum_{i=1}^{M} \langle x, \varphi_i \rangle \psi_i$$

where $(\psi_i)_{i=1}^{M}$ is a dual frame of $(\varphi_i)_{i=1}^{M}$.

Riesz Basis

If we know $(\varphi_i)_{i=1}^{M}$ is a Riesz basis, then there is only unique dual frame.

Pf: From the dual frame formula,

$$\Phi = \Phi_0 + U$$

and $$\Phi U^* = 0$$

But Φ is a basis, $\ker \Phi = \{0\}$. $\square$

Indeed, $$x = \sum_{i=1}^{M} \langle x, \varphi_i \rangle S^{-1} \varphi_i = \sum_{i=1}^{M} \langle x, S^{-1} \varphi_i \rangle \varphi_i$$

<u>Definition</u>: We say that (φ_i) and (ψ_j) is a biorthogonal sequence if

$$\langle \varphi_i, \psi_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = \delta_{ij}$$

<u>Proposition</u>: $(\varphi_i)_{i=1}^N$ is a Riesz basis, then $(\varphi_i)_{i=1}^N$ and $(S^{-1}\varphi_i)_{i=1}^N$ forms a biorthogonal sequence.

<u>P.f.</u> For a Riesz basis,

$$x = \sum_{i=1}^N \langle x, S^{-1}\varphi_i \rangle \varphi_i$$

Take $x = \varphi_j$

$$\varphi_j = \sum_{i=1}^N \langle \varphi_j, S^{-1}\varphi_i \rangle \varphi_i$$

Basis $\Rightarrow$ $\langle \varphi_j, S^{-1}\varphi_i \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

$\square$

Riesz basis, when one $\langle x, \varphi_i \rangle$ is gone, there is no way to reconstruct.

Frame with fixed erasure positions

Now suppose that a frame $(\varphi_i)_{i=1}^M$ is given, Then there are more than one representation in (φ_i)

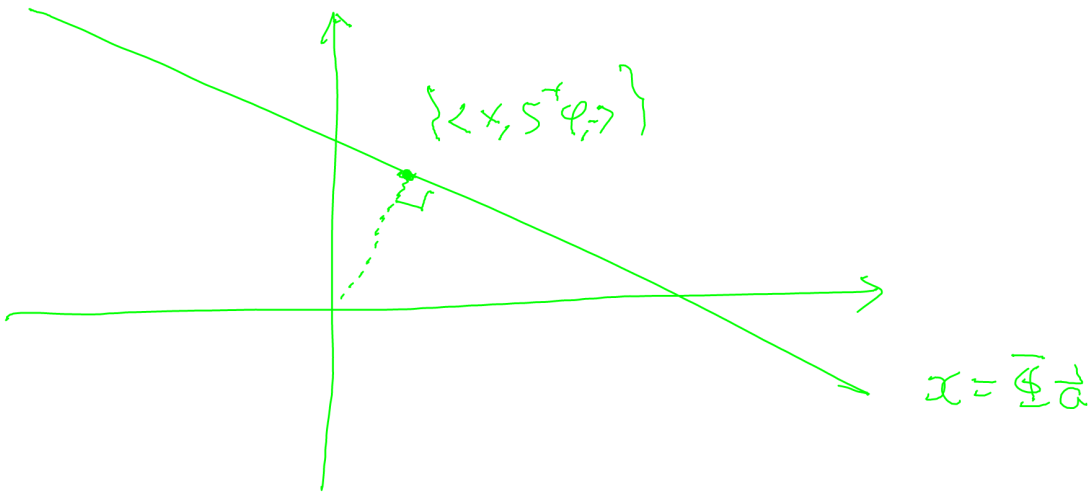
$$x = \sum_{i=1}^M a_i \varphi_i = \sum_{i=1}^M \langle x, S^{-1}\varphi_i \rangle \varphi_i$$

The set of all possible representation is an affine plane.

$$\left\{ \vec{a} : \quad x = \Phi \vec{a} \right\}$$

And $\vec{a} = \{\langle x, S^{-1}\varphi_i \rangle\}_{i=1}^M$ is the least square solution, since we proved

$$(*) \quad \sum_{i=1}^M |a_i|^2 = \sum_{i=1}^M |\langle x, S^{-1}\varphi_i \rangle|^2 + \sum_{i=1}^M |a_i - \langle x, S^{-1}\varphi_i \rangle|^2$$



<u>Theorem</u>: Let $(\varphi_k)_k$ be a frame

(i) If $\langle \varphi_j, S^{-1}\varphi_j \rangle \neq 1$, then

$\{\varphi_k\}_{k\neq j}$ is a frame

(ii) If $\langle \varphi_j, S^{-1}\varphi_j \rangle = 1$, then

$\{\varphi_k\}_{k\neq j}$ is not complete

(or $\operatorname{span}\{\varphi_k\}_{k\neq j} \neq \mathcal{H}$)

(Ref: Christensen Theorem 5.4.7.)

Pf. We will use the pyth. relation (*).

$$\varphi_j = \sum \langle \varphi_j, S^{-1}\varphi_k \rangle \varphi_k .$$

$$1 = \sum_{k=1}^{\infty} |\langle \varphi_j, S^{-1}\varphi_k \rangle|^2 + |1 - \langle \varphi_j, S^{-1}\varphi_j \rangle|^2 + \sum_{k\neq j} |\langle \varphi_j, S^{-1}\varphi_k \rangle|^2$$

$$= |\langle \varphi_j, S^{-1}\varphi_j \rangle|^2 + |1 - \langle \varphi_j, S^{-1}\varphi_j \rangle|^2$$

$$+ 2 \sum_{k\neq j} |\langle \varphi_j, S^{-1}\varphi_k \rangle|^2 .$$

(**)

(b) If $\langle \varphi_j, S^{-1}\varphi_j \rangle = 1$.

Then $$\sum_{k \neq j} |\langle \varphi_j, S^{-1}\varphi_k \rangle|^2 = 0$$

$$\Rightarrow \sum_{k \neq j} (\langle S^{-1}\varphi_j, \varphi_k \rangle)^2 = 0$$

$$\Rightarrow \langle S^{-1}\varphi_j, \varphi_k \rangle = 0 \quad \forall j \neq k$$

Hence $S^{-1}\varphi_j$ is orthogonal to all φ_k, $k \neq j$.

This shows that $\operatorname{Span}\{\varphi_k\} \neq \mathcal{H}^N$.

(a) If $\langle \varphi_j, S^{-1}\varphi_j \rangle \neq 1$. Then

(**) shows that $\langle \varphi_j, S^{-1}\varphi_j \rangle < 1$.

Then let $a_k = \langle \varphi_j, S^{-1}\varphi_k \rangle$

$$\varphi_j = \sum_{j=1}^{\infty} a_j \varphi_j \Rightarrow \varphi_j = \frac{1}{1-a_j} \sum_{k \neq j} a_k \varphi_k$$

(Finite dimension) $\Rightarrow$ $\operatorname{Span}\{\varphi_k\} = \operatorname{Span}\{\varphi_k\}_{k \neq j}$.

$\Rightarrow$ $\{\varphi_k\}_{k \neq j}$ is still a spanning set and hence a frame.

(Infinite dimension) We need some extra work

$\forall \varphi \in \mathcal{H}$, $\quad \langle \varphi, \varphi_j \rangle \doteq \frac{1}{1-a_j} \sum_{k \neq j} a_k \langle \varphi, \varphi_k \rangle$

$$|\langle \varphi, \varphi_j \rangle|^2 \leq \frac{1}{|1-a_j|^2} \left(\sum_{k \neq j} |a_k|^2 \right) \left(\sum_{k \neq j} |\langle \varphi, \varphi_k \rangle|^2 \right)$$

$$:= C \cdot \sum_{k \neq j} |\langle \varphi, \varphi_k \rangle|^2$$

Thus $\quad A\|\varphi\|^2 \leq \sum_{k=1}^{\infty} |\langle \varphi, \varphi_k \rangle|^2$

$$\leq (C+1) \sum_{k \neq j} |\langle \varphi, \varphi_k \rangle|^2$$

$$\Rightarrow \quad \frac{A}{1+C} \|\varphi\|^2 \leq \sum_{k \neq j} |\langle \varphi, \varphi_k \rangle|^2$$

This shows that $\{\varphi_k\}_{k \neq j}$ has the lower frame bound.

The upper frame bound is automatically true. $\square$

<u>Remark</u>: This proof shows also that the removal of one vector leaves either a frame or incomplete set.

Frames without fixed erasure position

(c.f. Optimal dual frames for erasures by Lopez, Han)

In practice, every time the signal arrived will have variable erasure position. In this case, we are unable to design or redesign a dual frame to suit our need.

The problem now is to choose a suitable dual frame / frame to minimize the errors arised. Let

$$\mathcal{D}_m = \left\{ D \in M \times M : \begin{array}{l} D \text{ is diagonal with } m \text{ zeros} \\ \text{and } M-m \text{ non-zeros} \end{array} \right\}$$

e.g. $$\mathcal{D}_1 = \left\{ \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \cdots \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} \right\}$$

$$\# \mathcal{D}_m = \binom{M}{m}$$

We consider there are m erasures.

The 1s are the erasure positions.

Given a dual frame pair. (Φ, Ψ).

Suppose that erasure position are

$$J = \{ j_1, \cdots j_m \}.$$

We will approximate x by

$$\tilde{x} = \sum_{j \notin J} \langle x, \varphi_j \rangle \psi_j$$

the errors is given by

$$x - \tilde{x} = \sum_{j \in J} \langle x, \varphi_j \rangle \psi_j .$$

$$= \Psi D_J \Phi^* x .$$

where $D_J \in \mathcal{D}_m$ with 1 at the j_i diagonal. position. Define the error operator for J

$$E_J x = \Psi D_J \Phi^* x .$$

Then we want to see the maximum possible errors

$$d_m(\Phi \Psi) = \max \{ \| \Psi D_J \Phi^* \| : D_J \in \mathcal{D}_m \}$$

Ambitious Goal : Given $\mathcal{H}^N$ and M. The optimal dual frame pair :

$$(\Phi_0, \Psi_0) = \arg\min d_m(\Phi, \Psi). \ ?$$

As this is certainly too ambitious, we consider some other simpler problem.

If Φ is a Parseval frame and Φ is the canonical dual.

$$\mathcal{E}_m(N,M) = \min\{ d_m(\Phi,\Phi) : \Phi \text{ is Parseval}\}$$

<u>Theorem</u> (Holme and Paulsen)

$\mathcal{E}_1(N,M) = \frac{N}{M}$ and it attains whenever Φ is a Parseval frame and $\|\varphi_i\| = C$ (Equal norm Parseval frame)

P.f.: Suppose the erasured position is at p.

$$\Phi D_p \Phi^* x = \langle x, \varphi_p \rangle \varphi_p$$

Thus $\|\Phi D_p \Phi^* x\| = |\langle x, \varphi_p \rangle| \, \|\varphi_p\|$

$$\leq \|\varphi_p\|^2 \|x\|.$$

We can show that the norm is attained

$$\|\Phi D_p \Phi^*\| = \|\varphi_p\|^2.$$

Hence $d_1(\Phi, \Phi) = \max\{\|\varphi_p\|^2 : p = 1, 2, \cdots M\}$

Note that

$$N = \mathrm{Tr}(\Phi\Phi^*) = \mathrm{Tr}(\Phi^*\Phi)$$

$$= \sum_{p=1}^{M} \|\varphi_p\|^2$$

$$\leq M \cdot d_1(\Phi, \Phi)$$

Thus $d_1(\Phi, \Phi) \geq \frac{N}{M}$.

Equality attained $\Leftrightarrow$ $\|\varphi_p\|^2 = d_1(\Phi, \Phi)$

$\Leftrightarrow$ Equal norm $\square$.

Another Question:

Fix a frame Φ, we say a dual frame Ψ_0 is an <u>optimal dual frame for 1-erasure</u> if

$$d_1(\Phi, \Psi_0) = \min\{d_1(\Phi, \Psi) : \Psi, \text{ is a dual frame for } \Phi\}$$

Ψ_0 is an optimal dual frame for m erasure if it is optimal for $(m-1)$ and

$$d_m(\Phi, \Psi_0) = \min \left\{ d_m(\Phi, \Psi) : \Psi \text{ is a dual frame for } \Phi \right\}$$

Given Φ, study the structure of the duals Ψ optimal for m-erasures.

(Student Presentation topic !)

Full - Spark frame

The best frame that is the most resilient to erasure should be frames that remain frames upon erasures until it becomes a basis.

Definition: Let $(\varphi_i)_{i=1}^{M}$ be a frame on $\mathcal{H}^N$. It is called a full-spark frame if

if the erasure of any $M-N$ vectors leaves a frame. i.e. For any $I \subset \{1, \dots M\}$. $|I| = M-N$ the sequence $(\varphi_i)_{i=1}^{M}, i \notin I$ is still a frame for $\mathcal{H}^N$.

A frame $\{\varphi_i\}_{i=1}^{M}$ has a <u>uniform m-excess</u> if any removal of m vectors leaves a frame.

<u>Proposition</u> $(\varphi_i)_{i=1}^{M}$ is a full-spark frame if and only if any N columns of Φ is linearly independent.

It should be expected that frames with uniform excess, there should be chances to recover the lost coefficient. (Student Presentation)