Lecture 4: Reconstruction with erasure.

When we received the data \( \{ x, \varphi_i \}_{i=1}^m \) without any loss, then

\[
x = \sum_{i=1}^m < x, \varphi_i > \varphi_i
\]

where \( \{ \varphi_i \}_{i=1}^m \) is a dual frame of \( \{ \varphi_i \}_{i=1}^m \).

\textbf{Riesz Basis}

If we know \( \{ \varphi_i \}_{i=1}^m \) is a Riesz basis, then there is only a unique dual frame.

\textbf{Proof:} From the dual frame formula,

\[
\mathcal{E} = \mathcal{E}_0 + \mathcal{U}
\]

and \( \mathcal{E} \mathcal{U}^* = \mathcal{0} \)

But \( \mathcal{E} \) is a basis, \( \ker \mathcal{E} = \{ 0 \} \).

Indeed,

\[
x = \sum_{i=1}^m < x, \varphi_i > \varphi_i = \sum_{i=1}^m < x, S \varphi_i > \varphi_i
\]
Definition: We say that \((\psi_i)\) and \((\phi_j)\) is a biorthogonal sequence if
\[
\langle \psi_i, \phi_j \rangle = \begin{cases} 1 & i = j \\ 0 & \text{ if } i \neq j \end{cases}
\]

Proposition: \((\psi_i)_{i=1}^N\) is a Riesz basis, then \((\phi_j)_{j=1}^N\) and \((S^\dagger \psi_i)_{i=1}^N\) forms a biorthogonal sequence.

P.f. For a Riesz basis,
\[
\chi = \sum_{i=1}^N <x, S^\dagger \psi_i> \psi_i
\]

Take \(\chi = \phi_j\)
\[
\psi_j = \sum_{i=1}^N <\phi_j, S^\dagger \psi_i> \psi_i
\]

Riesz basis, when one \(\langle x, \psi_i \rangle\) is gone, there is no way to reconstruct.
Frame with fixed erasure positions

Now suppose that a frame \((\varphi_i)_{i=1}^m\) is given. Then there are more than one representation in \((\varphi_i)\)

\[
x = \sum_{i=1}^m a_i \varphi_i = \sum_{i=1}^m (x, S_i^* \varphi_i) \varphi_i
\]

The set of all possible representation is an affine plane.

\[
\left\{ \hat{a} : \; \chi = \Phi \hat{a} \right\}
\]

And \(\hat{a} = [(<x, S_i^* \varphi_i>)_{i=1}^m]\) is the least square solution, since we proved

\[
(\star) \; \sum_{i=1}^m |a_i|^2 = \sum_{i=1}^m \left( <x, S_i^* \varphi_i> \right)^2 + \sum_{i=1}^m \left( a_i - <x, S_i^* \varphi_i> \right)^2
\]
Theorem: Let \((\psi_k)\) be a frame

(i) If \(\langle \psi_j, S^*\psi_j \rangle \neq 1\), then

\[\{\psi_k\}_{k \neq j} \text{ is a frame}\]

(ii) If \(\langle \psi_j, S^*\psi_j \rangle = 1\), then

\[\{\psi_k\}_{k \neq j} \text{ is not complete}\]

\(\text{(or \\text{Span } \{\psi_k\}_{k \neq j} \neq H)}\)

(Ref: Christensen, Theorem 5.4.7)

\[\text{Proof: We will use the pythagorean relation } (**).\]

\[\psi_j = \sum_{k \neq j} \langle \psi_j, S^*\psi_k \rangle \psi_k.\]

\[1 = \sum_{k=1}^{\infty} |\langle \psi_j, S^*\psi_k \rangle|^2 + |1 - \langle \psi_j, S^*\psi_j \rangle|^2 + \sum_{k \neq j} |\langle \psi_j, S^*\psi_k \rangle|^2 \]

\[= |\langle \psi_j, S^*\psi_j \rangle|^2 + |1 - \langle \psi_j, S^*\psi_j \rangle|^2 \]

\[+ 2 \sum_{k \neq j} |\langle \psi_j, S^*\psi_k \rangle|^2.\]

\((***)\)
(b) If \( \langle \varphi_j, S^T \varphi_j \rangle = 1 \).

Then \( \sum_{k \neq j} \left| \langle \varphi_j, S^T \varphi_k \rangle \right|^2 = 0 \)

\[ \Rightarrow \sum_{k \neq j} \left( \langle S^T \varphi_j, \varphi_k \rangle \right)^2 = 0 \]

\[ \Rightarrow \langle S^T \varphi_j, \varphi_k \rangle = 0 \quad \forall j \neq k \]

Hence \( S^T \varphi_j \) is orthogonal to all \( \varphi_k \), \( k \neq j \).

This shows that \( \text{Span} \{ \varphi_k \} \neq H^N \).

(a) If \( \langle \varphi_j, S^T \varphi_j \rangle \neq 1 \). Then

\((***)\) shows that \( \langle \varphi_j, S^T \varphi_j \rangle < 1 \).

Then let \( \alpha_k = \langle \varphi_j, S^T \varphi_k \rangle \)

\[ \varphi_j = \sum_{j=1}^{\infty} \alpha_j \varphi_j \quad \Rightarrow \quad \varphi_j = \frac{1}{\lambda_j} \sum_{k \neq j} \alpha_k \varphi_k \]

\((\text{Finite Dimension})\) \Rightarrow \text{Span} \{ \varphi_j \} = \text{Span} \{ \varphi_k \}_{k \neq j} \).

\( \Rightarrow \{ \varphi_k \}_{k \neq j} \) is still a spanning set and hence a frame.
(Infinite dimension) We need some extra work

\[ A \psi \in \mathcal{H}, \quad \langle \psi, \psi_j \rangle = \frac{1}{\| \psi_j \|^2} \sum_{k \neq j} a_k \langle \psi, \phi_k \rangle \]

\[ |\langle \psi, \psi_j \rangle|^2 \leq \frac{1}{1 - \| \psi \|^2} \left( \sum_{k \neq j} |a_k|^2 \right) \left( \sum_{k \neq j} |\langle \psi, \phi_k \rangle|^2 \right) \]

\[ : = C \cdot \sum_{k \neq j} |\langle \psi, \phi_k \rangle|^2 \]

Thus

\[ \| \psi \|^2 \leq \sum_{k=1}^{\infty} |\langle \psi, \phi_k \rangle|^2 \]

\[ \leq (C + 1) \sum_{k \neq j} |\langle \psi, \phi_k \rangle|^2 \]

\[ \Rightarrow \frac{A}{1 + C} \| \psi \|^2 \leq \sum_{k \neq j} |\langle \psi, \phi_k \rangle|^2 \]

This shows that \( \{ \phi_k \}_{k \neq j} \) has the lower frame bound.

The upper frame bound is automatically true.

\[ \square \]

Remark: This proof shows also that the removal of one vector leaves either a frame or incomplete set.
Frames without fixed erasure position

c.f. Optimal dual frames for erasures by Lopez, Han

In practice, every time the signal arrived will have variable erasure position. In this case, we are unable to design or redesign a dual frame to suit our need.

The problem now is to choose a suitable dual frame / frame to minimize the errors incurred. Let

$$\mathcal{D}_m = \{ D \in M \times M : D \text{ is diagonal with } m \text{ zens and } M - m \text{ non-zeros} \}$$

e.g. $$\mathcal{D}_1 = \{ \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 0 & \cdots \\ \vdots & \ddots \\ 1 & \cdots \end{bmatrix}, \ldots, \begin{bmatrix} 1 & \cdots \end{bmatrix} \}$$

$$\# \mathcal{D}_m = \binom{M}{m}$$

We consider there are $$m$$ erasures.

The 1s are the erasure positions.
Given a dual frame pair \((\Phi, \Psi)\).

Suppose that erasure position are
\[ J = \{ j_1, \ldots, j_m \} \]
we will approximate \(x\) by
\[ \overline{x} = \sum_{j \notin J} <x, \phi_j> \psi_j \]

the error is given by
\[ x - \overline{x} = \sum_{j \in J} <x, \phi_j> \psi_j \]

\[ = \overline{\Phi} D_J \overline{\Psi}^* x. \]

where \( D_J \in \mathbb{D}_m \) with 1 at the \( j_i \) diagonal position. Define the error operator for \( J \)
\[ E_J x = \overline{\Phi} D_J \overline{\Psi}^* x. \]

Then we want to see the maximum possible errors
\[ d_m(\Phi, \Psi) = \max \{ || \overline{\Phi} D_J \overline{\Psi}^* || : D_J \in \mathbb{D}_m \} \]

Ambitious Goal: Given \( N \) and \( M \). The optimal dual frame pair:
\[ (\Phi_0, \Psi_0) = \text{argmin} \ d_m(\Phi, \Psi). \]
As this is certainly too ambitious, we
consider some other simpler problem.
If $\Phi$ is a Parseval frame and $\Phi^*$ is the
canonical dual,

$$E_{\min}(N,M) = \min \left\{ d_m(\Phi,\Phi^*) : \Phi^* \text{ is Parseval} \right\}$$

**Theorem** (Holme and Paulsen)

$$E_{\min}(N,M) = \frac{N}{M}$$ and it attains whenever

$\Phi$ is a Parseval frame and $\|\psi\| = c$  

(Equal norm Parseval frame)

**P.f.** Suppose the erased position is at $p$.

$$\Phi^* D_p \Phi x = \langle x, \psi_p \rangle \psi_p$$

Thus

$$\| \Phi^* D_p \Phi x \| = |\langle x, \psi_p \rangle| \| \psi_p \|$$

$$\leq \| \psi_p \|^2 \| x \|.$$  

We can show that the norm is attained

$$\| \Phi^* D_p \Phi x \| = \| \psi_p \|^2.$$
Here \( d_1(\Phi, \Phi) = \max \left\{ \| \varphi_p \|^2 : p = 1, 2, \ldots, M \right\} \)

Note that

\[
N = \text{Tr} (\Phi^* \Phi^*) = \text{Tr} (\Phi^* \Phi) = \sum_{p=1}^{M} \| \varphi_p \|^2 \\
\leq M \cdot d_1(\Phi, \Phi)
\]

Thus \( d_2(\Phi, \Phi) \geq \frac{N}{M} \).

Equality attained \( \iff \| \varphi_p \|^2 = d_1(\Phi, \Phi) \)

\( \iff \text{Equal norm} \) \( \square \).

Another Question:

Fix a frame \( \Phi \), we say a dual frame \( \Phi_0 \) is an optimal dual frame for 1-erasure if

\[
d_1(\Phi, \Phi_0) = \min \{ d_1(\Phi, \Phi') : \Phi, \text{is a dual frame for } \Phi \}
\]
It is an optimal dual frame for m-erasures if it is optimal for (m-1) and
\[ d_m(\mathcal{E}_0) = \min \{ d_m(\mathcal{U}^\perp) : \mathcal{U}^\perp \text{ is a dual frame} \} \]
for any \( \mathcal{U}^\perp \).

Given \( \mathcal{E} \), study the structure of the duals of \( \mathcal{E} \) optimal for m-erasures.

(Student Presentation topic!)

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**Full-Spark Frame**

The best frame that is the most resilient to erasure should be frames that remain frames upon erasures until it becomes a basis.

**Definition:** Let \( \{ \psi_i \}_{i=1}^N \) be a frame on \( \mathbb{F}^N \). It is called a full-spark frame if
If the erasure of any $M-N$ vectors leaves a frame, i.e., for any $I \subseteq \{1, \ldots, M\}$, $|I| = M-N$, the sequence $(\psi_i)_{i=1}^M$, $i \notin I$ is still a frame for $\mathbb{R}^N$.

A frame $(\psi_i)_{i=1}^M$ has a uniform $m$-excess if any removal of $m$ vectors leaves a frame.

**Proposition** $(\psi_i)_{i=1}^M$ is a full-spark frame if and only if any $N$ columns of $\mathbb{F}$ is linearly independent.

It should be expected that frames with uniform excess, there should be chances to recover the lost coefficient. (Student Presentation)