In this course, we will be discussing around the ideas of frames on finite and infinite dimensional vector spaces. This has been found extremely useful in applied analysis.

Motivation

Let $\mathbf{x}$ be a data. It means $\mathbf{x}$ is a vector in $\mathbb{R}^N$ or $\mathbb{C}^N$. We denote it by $\mathbb{H}^N$, ($\mathbb{H} = \mathbb{R}$ or $\mathbb{C}$).

On $\mathbb{H}^N$, we have a representation system $(\{\psi_i\})_i$, which span $(\mathcal{D}) = \mathbb{H}^N$.

By data processing / signal processing, we mean we do a decomposition of $\mathbf{x}$.

$$\mathbf{x} \quad \rightarrow \quad (\langle \mathbf{x}, \psi_i \rangle)^M_{i=1}$$
What is the error $\| \mathbf{x} - \mathbf{x}' \|$?

**Example 1:** $\{ \mathbf{\phi}_i \}_{i=1}^N$ is an orthonormal basis for $\mathbb{H}^N$.

Assume there is no corruption,

Reconstruction is very easy:

\[
\mathbf{x} = \sum_{i=1}^{N} \langle \mathbf{x}, \mathbf{\phi}_i \rangle \mathbf{\phi}_i.
\]
However, life is not easy.

Corruption is expected!!

E.g. One of the coefficients is lost.

It will not be constructible.

**Redundancy takes the advantage!**

If one of the $\langle \vec{x}, \vec{\psi}_i \rangle$ is lost, we should expect $\{ \vec{\psi}_i \}$ is still a spanning set if it is redundant.

**Question:** How to reconstruct $\vec{x}$ from $\{ \langle \vec{x}, \vec{\psi}_i \rangle \}$?

$$\vec{x} = \sum_{i=1}^{M} \langle \vec{x}, \vec{\psi}_i \rangle \vec{\psi}_i$$

What happens if the data is corrupted?
§1 Frames

Definition of frames

A family of vectors \( \{ \psi_i \}_{i \in I} \) is called a frame for a Hilbert space \( \mathcal{H} \) if there exists constants \( 0 < A \leq B < +\infty \) such that

\[
A \| x \|^2 \leq \sum_{i \in I} | \langle x, \psi_i \rangle |^2 \leq B \| x \|^2 \quad \forall x \in \mathcal{H}.
\]

1. If \( \dim \mathcal{H} < +\infty \), then \( |I| < +\infty \), \( \{ \psi_i \}_{i=1}^{M} \) is called a finite frame.

2. \( A, B \) are called lower and upper frame bounds. The optimal frame bounds are denoted as \( A_{opt}, B_{opt} \).
3) If \( \{ \psi_i \}_{i \in I} \) satisfies the upper bound, then it is called **Bessel sequence**.

4) If \( A = B \), \( \{ \psi_i \}_{i \in I} \) is called a **tight frame**.

5) If \( A = B = 1 \), \( \{ \psi_i \}_{i \in I} \) is called a **Parseval frame**.

**Theorem (All spanning sets are frames)**

If \( \{ \psi_i \}_{i \in I} \) is a frame for \( \mathbb{H}^N \) if and only if

\[
\text{span} \{ \psi_i \} = \mathbb{H}^N
\]

**Proof:** \( (\Rightarrow) \) If \( \text{span} \{ \psi_i \} \neq \mathbb{H}^N \), then there exists \( x \) s.t.

\[
x \perp \text{span} \{ \psi_i \}
\]

i.e. \( \langle x, \psi_i \rangle = 0 \) \( \forall i \).
From the lower bound

\[ A \|x\| = \sum_{i=1}^{M} \langle x, \tilde{\psi}_i \rangle^2 = 0 \]

\[ \Rightarrow \|x\| = 0. \text{ and hence } x = 0. \]

Thus \( \text{span} \{ \psi_i \} = \{0\} \) and \( \text{span} \{ \psi_i \} = H^N \)

\((\Leftarrow)\) Conversely, if \( \text{span} \{ \psi_i \} = H^N \),

but frame inequality fails.

Note that the lower bounds means

\[ A \leq \sum_{i=1}^{M} \left| \langle \frac{x}{\|x\|}, \tilde{\psi}_i \rangle \right|^2. \]

i.e.

\[ A \leq \sum_{i=1}^{M} \left| \langle x, \psi_i \rangle \right|^2 \]

for all \( x \) in the unit ball of \( H^N \).

Now lower bound fails implies

there exists a sequence \( x_1, x_2, \ldots x_n \) in the unit ball such that

\[ \frac{1}{M} \sum_{i=1}^{M} \left| \langle x_n, \psi_i \rangle \right|^2 < \frac{1}{n}. \]
But unit ball is compact, we can find a subsequence $x_{n_k}$ such that $x_{n_k} \to x^*$

$$\lim_{k \to \infty} \sum_{i=1}^{m} |\langle x_{n_k}, \psi_i \rangle|^2 < \lim_{k \to \infty} \frac{1}{\eta_k}$$

$$\Rightarrow \sum_{i=1}^{m} |\langle x^*, \psi_i \rangle|^2 = 0$$

$$\Rightarrow |\langle x^*, \psi_i \rangle|^2 = 0$$

$$\Rightarrow x^* \perp \text{span} \{ \psi_i \} \quad \text{and} \quad \|x^*\| = 1$$

This contradicts $\text{span} \{ \psi_i \} = \mathbb{H}^N$. \[\square\]

Hence, essentially everything are "frames"!!

We will see $\frac{B}{A}$ determines the performance of a frame.

$\frac{B}{A}$ is called the condition number.
An orthonormal basis is clearly a Parseval frame. However, the converse is not true.

E.g., if \( \{ e_n \}_{n=1}^\infty \) and \( \{ g_n \}_{n=1}^\infty \) are orthonormal bases for \( l^2 \mathbb{N} \). Then

\[
\sum_{n=1}^\infty | \langle x, e_n \rangle |^2 = \| x \|^2
\]

\[
\sum_{n=1}^\infty | \langle x, g_n \rangle |^2 = \| x \|^2
\]

\[
\Rightarrow \sum_{n=1}^\infty | \langle x, e_n \rangle |^2 + \sum_{n=1}^\infty | \langle x, g_n \rangle |^2 = 2 \| x \|^2.
\]

Thus \( \{ e_n \} \cup \{ g_n \} \) forms a tight frame.

Also

\[
\sum_{n=1}^\infty | \langle x, \frac{1}{\sqrt{2}} e_n \rangle |^2 + \sum_{n=1}^\infty | \langle x, \frac{1}{\sqrt{2}} g_n \rangle |^2 = \| x \|^2
\]

Thus \( \{ \frac{1}{\sqrt{2}} e_n \} \cup \{ \frac{1}{\sqrt{2}} g_n \} \) forms a Parseval frame.
Proposition: A Parseval frame \( \{ \psi_n \}_{n=0}^M \) is an orthonormal basis if and only if \( \| \psi_n \| = 1 \) \( \forall n \).

Proof:
\[
\sum_{n=1}^{M} |<x, \psi_n>|^2 = \|x\|^2
\]

Take \( x = \psi_n_0 \), then
\[
|<x, \psi_n_0>|^2 = |<\psi_n_0, \psi_n_0>|^2 = \|\psi_n_0\|^4.
\]

Thus,
\[
\|\psi_n_0\|^4 + \sum_{n \neq n_0} |<\psi_n_0, \psi_n>|^2 = \|\psi_n_0\|^2.
\]

\( \Rightarrow \) If \( \{ \psi_n \}_{n=1}^M \) is O.N.B., then
\[
|<\psi_n_0, \psi_n>| = 0 \quad \forall n \neq n_0
\]

\( \Rightarrow \) \( \|\psi_n_0\|^4 = \|\psi_n_0\|^2 \)

\( \Rightarrow \) \( \|\psi_n_0\| = 1 \).

\( \Leftarrow \) \( \|\psi_n_0\| = 1 \) \( \Rightarrow \) \( \sum_{n \neq n_0} |<\psi_n_0, \psi_n>|^2 = 0 \)

\( \Rightarrow \) \( <\psi_n_0, \psi_n> = 0 \quad \forall n \).
Non trivial example of tight frame

E.g. \( \{ \begin{bmatrix} i \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \end{bmatrix} \} \)

is a tight frame of frame constant \( \frac{3}{2} \).

Proof: 
1. Compute by brute force
2. Beautiful theory

coming soon !!
More examples

1. If \( \{ e_n \} \) is an orthonormal basis,
\[
\{ e_1, e_2, e_3, \ldots, e_n, e_n \}
\]
is a tight frame. The frame bound is 2.

2. In general, union of \( L \) orthonormal bases is a tight frame with frame bound \( L \).

3. An important orthonormal basis

Let \( M \in \mathbb{N} \). \( \omega = \frac{2\pi}{M} \).

Then the Discrete Fourier Transform (DFT) matrix / Fourier Matrix.

\[
D_M = \frac{1}{\sqrt{M}} \left[ \omega^{jk} \right]_{j,k=0}^{M-1}
\]

E.g.,

\( M=2 \)

\[
D_2 = \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right] = \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right]
\]
\[
D_3 = \frac{1}{\sqrt{3}} \begin{bmatrix}
1 & 1 & 1 \\
e^{\frac{2\pi i}{3}} & e^{\frac{1}{3}} & e^{\frac{2\pi i}{3}} \\
e^{\frac{2\pi i}{3}} & e^{\frac{2\pi i}{3}} & e^{\frac{2\pi i}{3}} \\
e \end{bmatrix}, \quad w = e^{\frac{2\pi i}{3}}
\]

**Fact:** \( D_M \) is a unitary matrix.

Columns of \( D_M \) forms an orthonormal basis on \( \mathbb{C}^M \).

**P.f.:** \( D_M = \begin{bmatrix}
\frac{\sqrt{3}}{2} \\
1 \\
\frac{\sqrt{3}}{2}
\end{bmatrix} \)

\[
\frac{1}{\sqrt{3}} = \frac{1}{M} (\bar{w}^k) M \]

\[
\langle \bar{v}, \bar{w} \rangle = \sum_{k=1}^{M} \bar{w}^k \bar{w}^k = \sum_{k=1}^{M} \bar{w}^{(M-k)k} = \frac{\bar{w}^{(M-2)M} - 1}{\bar{w}^{2k} - 1} = 0 \quad (\bar{w} = 1)
\]
\[
\| \mathbf{W}_{ij} \| = \| \frac{1}{\sqrt{M}} (\mathbf{W}^k)^M_{k=1} \| \\
= \sum_{k=1}^{M} |W_{ik}|^2 \cdot \frac{1}{\sqrt{M}} \\
= 1.
\]

Thus they form an O.N.V.B.

A general construction of Tight frame.

Let \( \{ \psi_i \}_{i=1}^N \) be an orthonormal basis for \( \mathbb{C}^N \).

\[
\sum_{i=1}^{N} |\langle x, \psi_i \rangle|^2 = \| x \|^2 \quad \forall x \in \mathbb{C}^N.
\]

Let \( x = (x_1, \ldots, x_M, 0, 0 \ldots 0), \quad (M < N) \)

\[
\psi_{i}^{(M)} = (\psi_{i}^{(1)}, \ldots, \psi_{i}^{(M)}, 0, \ldots 0)
\]

Put \( x \) into \((\ast)\)

\[
\sum_{i=1}^{M} |\langle x, \psi_{i}^{(M)} \rangle|^2 = \| x \|^2.
\]
This implies \( \{ \psi_i^{(m)} \} \) forms a tight frame for \( C^M \).

**Remark:** We will see every tight frame are indeed projection of some orthonormal basis. \( \Box \)