In this course, we will be discussing around the ideas of frames on finite and infinite dimensional vector spaces. This has been found extremely useful in applied analysis

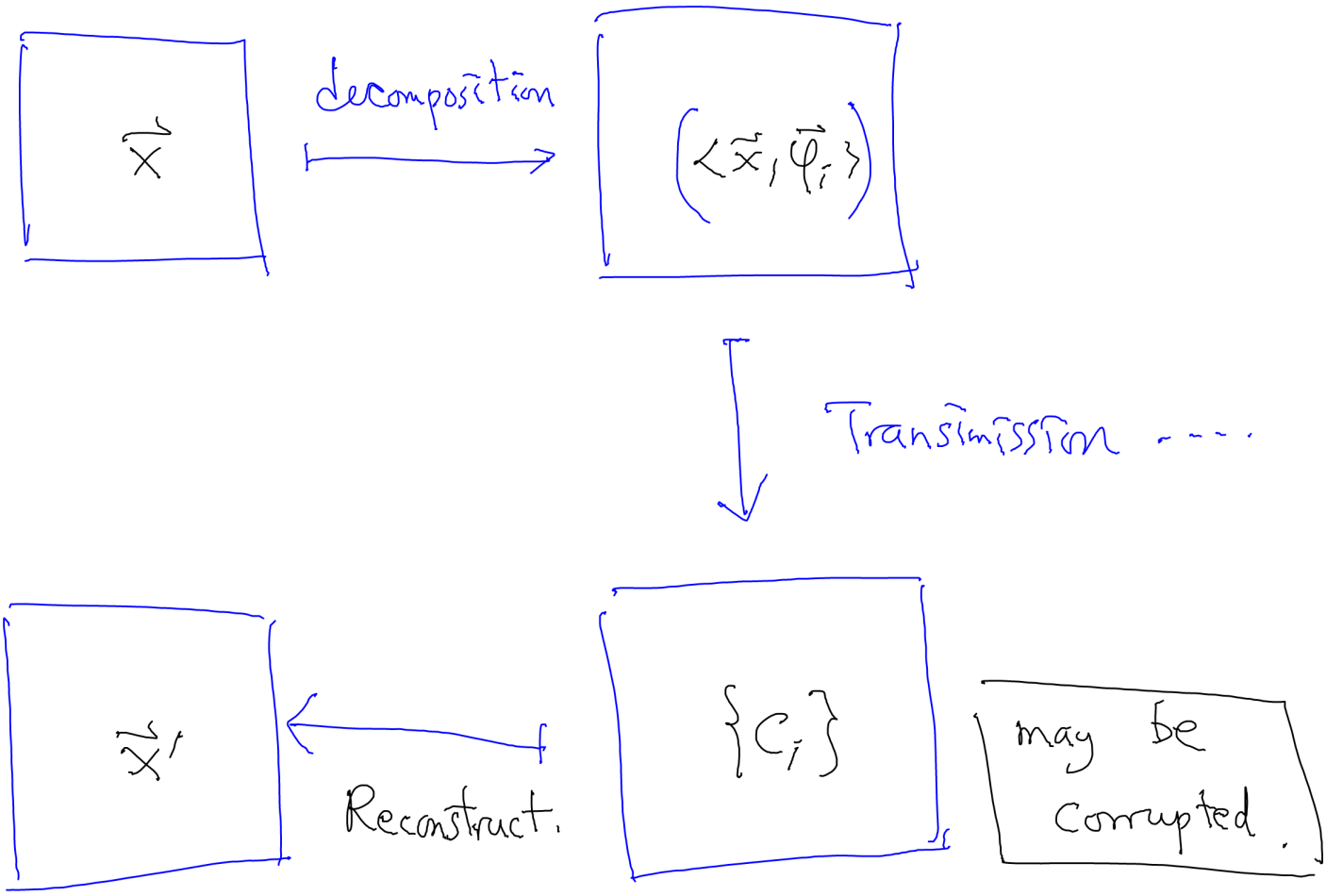
Motivation

Let $\vec{x}$ be a data. It means $\vec{x}$ is a vector in $\mathbb{R}^N$ or $\mathbb{C}^N$. We denote it by $\mathbb{H}^N$. $(\mathbb{H} = \mathbb{R}$ or $\mathbb{C})$

On $\mathbb{H}^N$, we have a representation system. $(\vec{\varphi}_i)_{i=1}^M$ which span $(\vec{\varphi}_i) = \mathbb{H}^N$.

By data Processing / signal Processing, we mean we do a decomposition of $\vec{x}$.

$$\vec{x} \longmapsto \left(\langle \vec{x}, \vec{\varphi}_i \rangle\right)_{i=1}^M$$



What is the error $\|\vec{x} - \vec{x}'\|$?

<u>Eg 1</u> : $\{\vec{\varphi}_i\}_{i=1}^{N}$ is an orthonormal basis for $\mathbb{H}^N$.

Assume there is no corruption,

Reconstruction is very easy :

$$\vec{x} = \sum_{i=1}^{N} \langle \vec{x}, \vec{\varphi}_i \rangle \vec{\varphi}_i .$$

However, life is not easy.

Corruption is expected !!

E.g. One of the coefficients is lost.

$\vec{x}$ will not be constructible.

Redundancy takes the advantage !

If one of the $\langle \vec{x}, \vec{\varphi}_{i_0} \rangle$ is lost, we should expect $\{\varphi_i\}_{i \neq i_0}$ is still a spanning set if it is redundant.

Question : How to reconstruct $\vec{x}$ from $\{\langle \vec{x}, \vec{\varphi}_i \rangle\}$?

$$\vec{x} \overset{?}{=} \sum_{i=1}^{m} \langle \vec{x}, \vec{\varphi}_i \rangle \vec{\psi}_i \quad ?$$

What happens if the data is corrupted ?

§1 Frames

Definition of frames

A family of vectors $\{\varphi_i\}_{i\in I}$ is called a frame for a Hilbert space $\mathcal{H}$ if there exists constants $0 < A \le B < +\infty$ such that

$$A\|x\|^2 \le \sum_{i\in I} |\langle x, \varphi_i\rangle|^2 \le B\|x\|^2 \quad \forall x \in \mathcal{H}.$$

① If $\dim \mathcal{H} < +\infty$, then $\# I < +\infty$, $\{\varphi_i\}_{i=1}^{M}$ is called a finite frame.

② . A, B are called lower and upper frame bounds. The optimal frame bounds are denoted as A_{opt} B_{opt}.

③ $\{\varphi_i\}_{i\in I}$ satisfies the upper bound, then it is called <u>Bessel sequence</u>

④ If $A=B$, $\{\varphi_i\}_{i\in I}$ is called a <u>tight frame</u>

⑤ If $A=B=1$, $\{\varphi_i\}_{i\in I}$ is called a <u>Parseval frame</u>.

<u>Theorem</u> (All spanning sets are frames)

$\{\varphi_i\}_{i=1}^{M}$ is a frame for $\mathcal{H}^N$ if and only if $\operatorname{span}\{\varphi_i\} = \mathcal{H}^N$

<u>Proof</u>: $(\Rightarrow)$ If $\operatorname{span}\{\varphi_i\} \neq \mathcal{H}^N$.

Then there exists $\vec{x}$ s.t.

$$\vec{x} \perp \operatorname{span}\{\varphi_i\}$$

i.e. $\langle \vec{x}, \vec{\varphi}_i \rangle = 0 \quad \forall i$.

From the lower bound

$$A\|\vec{x}\|^2 \leq \sum_{i=1}^{M} |\langle \vec{x}, \vec{\varphi}_i \rangle|^2 = 0$$

$\Rightarrow$ $\|\vec{x}\| = 0$. and hence $\vec{x} = 0$.

Thus $\text{span}\{\varphi_i\}^{\perp} = \{0\}$ and $\text{span}\{\varphi_i\} = \mathbb{H}^N$

($\Leftarrow$) Conversely, if $\text{span}\{\varphi_i\} = \mathbb{H}^N$,

but frame inequality fails.

Note that the lower bounds means

$$A \leq \sum_{i=1}^{M} \left| \left\langle \frac{x}{\|\vec{x}\|}, \vec{\varphi}_i \right\rangle \right|^2.$$

i.e.

$$A \leq \sum_{i=1}^{M} |\langle \hat{x}, \varphi_i \rangle|^2$$

for all $\hat{x}$ in the unit ball of $\mathbb{H}^N$.

Now lower bound fails implies

there exists a sequence $x_1, x_2, \cdots x_n$ in

the unit ball such that

$$\sum_{i=1}^{M} |\langle x_n, \varphi_i \rangle|^2 < \frac{1}{n}.$$

But <u>unit ball is compact</u>, we can find a subsequence of x_n such that

$$x_{n_k} \longrightarrow x^*$$

$$\lim_{k\to\infty} \sum_{i=1}^{m} |\langle x_{n_k}, \varphi_i \rangle|^2 < \lim_{k\to\infty} \frac{1}{n_k}$$

$$\Longrightarrow \quad \sum_{i=1}^{m} |\langle x^*, \varphi_i \rangle|^2 = 0$$

$$\Longrightarrow \quad \langle x^*, \varphi_i \rangle = 0$$

$$\Longrightarrow \quad x^* \perp \operatorname{span}\{\varphi_i\} \text{ and } \|x^*\| = 1$$

This contradicts $\operatorname{span}\{\varphi_i\} = \mathbb{H}^N$.

$\square$.

Hence, essentially everything are "frames" !!

We will see $\frac{B}{A}$ determines the performance of a frame.

B/A is called the <u>condition number</u>.

About Tight frame

An orthonormal basis is clearly a Parseval frame. However, the converse is not true.

<u>E.g.</u> If $\{e_n\}_{n=1}^N$ and $\{g_n\}_{n=1}^N$ are orthonormal basis for H^N. Then

$$\sum_{n=1}^N |\langle x, e_n\rangle|^2 = \|x\|^2$$

$$\sum_{n=1}^N |\langle x, g_n\rangle|^2 = \|x\|^2 .$$

$$\Rightarrow \quad \sum_{n=1}^N |\langle \tilde{x}, e_n\rangle|^2 + \sum_{n=1}^N |\langle \tilde{x}, g_n\rangle|^2 = 2\|x\|^2 .$$

Thus $\{e_n\} \cup \{g_n\}$ forms a tight frame.

Also $$\sum_{n=1}^N |\langle \tilde{x}, \tfrac{1}{\sqrt{2}} e_n\rangle|^2 + \sum_{n=1}^N |\langle \tilde{x}, \tfrac{1}{\sqrt{2}} g_n\rangle|^2 = \|x\|^2$$

Thus $\{\frac{1}{\sqrt{2}} e_n\} \cup \{\frac{1}{\sqrt{2}} g_n\}$ forms a Parseval frame

<u>Proposition</u> A Parseval frame $\{\varphi_n\}_{n=1}^{M}$ is an orthonormal basis if and only if $\|\varphi_n\| = 1$ $\forall$ n.

<u>Proof</u>: $$\sum_{n=1}^{M} |\langle x, \varphi_n \rangle|^2 = \|x\|^2$$

Take $x = \varphi_{n_0}$, then

$$|\langle x, \varphi_{n_0} \rangle|^2 = |\langle \varphi_{n_0}, \varphi_{n_0} \rangle|^2$$
$$= \|\varphi_{n_0}\|^4.$$

Thus, $$\|\varphi_{n_0}\|^4 + \sum_{n \neq n_0} |\langle \varphi_{n_0}, \varphi_n \rangle|^2 = \|\varphi_{n_0}\|^2$$

($\Rightarrow$) If $\{\varphi_n\}_{n=1}^{M}$ is O.N.B., then

$$\langle \varphi_{n_0}, \varphi_n \rangle = 0 \quad \forall n \neq n_0$$

$$\Rightarrow \quad \|\varphi_{n_0}\|^4 = \|\varphi_{n_0}\|^2$$

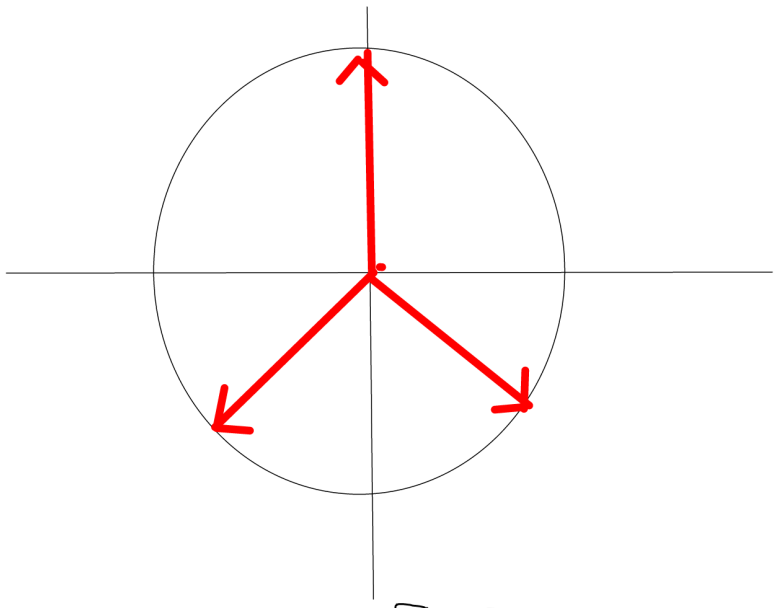
$$\Rightarrow \|\varphi_{n_0}\| = 1.$$

($\Leftarrow$) $\|\varphi_{n_0}\| = 1 \Rightarrow \sum_{n \neq n_0} |\langle \varphi_{n_0}, \varphi_n \rangle|^2 = 0$

$$\Rightarrow \langle \varphi_{n_0}, \varphi_n \rangle = 0 \quad \forall n.$$

Non trivial example of tight frame

E.g. (Mercedes - Benz frames).



$$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \sqrt{3}/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} -\sqrt{3}/2 \\ -1/2 \end{bmatrix} \right\}$$

is a tight frame of frame constant $\frac{3}{2}$.

<u>Proof</u>: ① Compute by brute force

② Beautiful theory

Coming soon !!

More examples

① If $\{e_n\}$ is an orthonormal basis,

$$\{e_1, e_1, e_2, e_2, \dots, e_n, e_n\}$$

is a tight frame. The frame bound is 2.

② In general, union of L orthonormal basis is a tight frame with frame bound L.

③ An important orthonormal basis

Let $M \in \mathbb{N}$. $\omega = e^{2\pi i \frac{1}{M}}$.

Then the Discrete Fourier Transform (DFT) matrix / Fourier Matrix.

$$D_M = \frac{1}{\sqrt{M}} \left[\omega^{jk} \right]_{j,k=0}^{M-1}$$

E.g.

$$M=2 \quad D_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & e^{2\pi i \frac{1 \cdot 1}{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$D_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i \frac{1\cdot 1}{3}} & e^{2\pi i \frac{1\cdot 2}{3}} \\ 1 & e^{2\pi i \frac{2\cdot 1}{3}} & e^{2\pi i \frac{2\cdot 2}{3}} \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}, \quad \omega = e^{2\pi i \frac{1}{3}}$$

<u>Fact</u>: D_M is a unitary matrix.

Columns of D_M forms an orthonormal basis on $\mathbb{C}^M$.

P.f:

$$D_M = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_M \\ | & & | \end{bmatrix}$$

$$\vec{v}_j = \frac{1}{\sqrt{M}} \left(\omega^{jk} \right)_{k=1}^{M}$$

$$\langle \vec{v}_j, \vec{v}_l \rangle = \sum_{k=1}^{M} \omega^{jk} \overline{\omega^{lk}}$$

$$= \sum_{k=1}^{M} \omega^{(j-l)k}$$

$$= \frac{\omega^{(j-l)\cdot M} - 1}{\omega^{j-l} - 1} = 0, \quad (\omega^M = 1)$$

$$\|\vec{U}_j\| = \left\| \frac{1}{\sqrt{M}} (\omega^{jk})_{k=1}^{M} \right\|$$

$$= \sum_{k=1}^{M} |\omega^{jk}|^2 \cdot \frac{1}{M}$$

$$= 1.$$

Thus they form an O.N.B.

<u>A general construction of Tight frame.</u>

Let $\{\varphi_i\}_{i=1}^{N}$ be an orthonormal basis for $\mathbb{C}^N$.

$$(*) \quad \sum_{i=1}^{N} |\langle x, \varphi_i \rangle|^2 = \|x\|^2 \quad \forall\, x \in \mathbb{C}^N.$$

Let $x = (x_1, \cdots x_M, 0, 0 \cdots 0)$, $(M < N)$

$$\varphi_i^{(M)} = (\varphi_i^{(1)}, \cdots \varphi_i^{(M)}, 0, \cdots 0)$$

Put x into $(*)$

$$\sum_{i=1}^{N} |\langle x, \varphi_i^{(M)} \rangle|^2 = \|x\|^2.$$

This implies $\{\varphi_i^{(M)}\}$ forms a tight frame for $\mathbb{C}^M$.

<u>Remark</u>: We will see every tight frame are indeed Projection of some orthonormal basis. $\star$