Lecture 12: Probabilistic RIP matrices.

Although deterministic construction of RIP matrices is hard to obtain, when we comes to random matrices, it exists in abundance.

Definition Let A be an $m \times N$ random matrix

a) If entries of A are independent random variable $\mathbb{P}(A_{ij} = 1) = \mathbb{P}(A_{ij} = -1) = \frac{1}{2}$

A is called Bernoulli matrix.

b). If A_{ij} are normal random variable $N(0,1)$

$$\mathbb{P}(|A_{ij}| \leq t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2}\, dx.$$

(A_{ij} independent)

c)
$$\mathbb{P}(|A_{ij}| \geq t) \leq \beta e^{-\kappa t^2}.$$

and A_{ij} independent and mean 0, variance 1.

Then A is subgaussian.

Bernoulli and Gaussian and subgaussian random matrices.

Gaussian $\Rightarrow$ Subgaussian

$$\mathbb{P}(|A_{ij}| \geq t) = \frac{2}{\sqrt{2\pi}} \int_t^\infty e^{-x^2/2} dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-(x+t)^2/2} dx$$

$$= \frac{2}{\sqrt{2\pi}} e^{-t^2/2} \int_0^\infty e^{-x^2/2 - \frac{tx}{2}} dx$$

$$\leq \frac{2}{\sqrt{2\pi}} e^{-t^2/2} \int_0^\infty e^{-x^2/2} dx.$$

$$\leq C \cdot e^{-t^2/2} \qquad \square$$

__Theorem__ Let A be an $m \times N$ subgaussian random matrix. Then there exists a constant $C > 0$ (depending on β, κ in subgaussian definition) such that the RIP constant δ_s of $\frac{1}{\sqrt{m}} A$ satisfies $\delta_s \le \delta$ with probability $\ge 1 - \varepsilon$

if
$$m \ge C \delta^{-2} \left(s \ln(eN/s) + \ln(2\varepsilon^{-1}) \right)$$

Main Ingredient of the proof.

(1). We say that $\vec{Y}$ is isotropic if
$$\mathbb{E} \, |\langle Y, x \rangle|^2 = \|x\|^2 .$$

(2). If for all $\vec{x} \in \mathbb{R}^N$ with $\|x\| = 1$,

$\langle Y, x \rangle$ is subgaussian, then

Y is called a subgaussian random vector.

Lemma: If $Y = (Y_1, \ldots Y_N)$ and Y_i are subgaussian with mean 0. Then Y is isotropic and subgaussian.

Pf.
$$\mathbb{E}\left[|\langle Y, x\rangle|^2\right] = \mathbb{E}\left[\sum_{i,j=1}^{N} x_i \bar{x}_j Y_i \bar{Y}_j\right]$$
$$= \sum_{i=1}^{N} \mathbb{E}\left[|x_i|^2 |Y_i|^2\right]$$
$$= \|x\|^2. \quad \left(\mathbb{E}|Y_i|^2 = 1\right)$$

Y is subgaussian. $\square$

Now
$$A = \begin{bmatrix} \text{---}\, Y_1 \,\text{---} \\ \vdots \\ \text{---}\, Y_m \,\text{---} \end{bmatrix}$$

$$\left| \left\| \frac{1}{\sqrt{m}} A x \right\|^2 - \|x\|^2 \right|$$
$$= \left| \frac{1}{m} \sum_{i=1}^{m} \left(|\langle Y_i, x\rangle|^2 - \|x\|^2 \right) \right|$$

$$= \left| \frac{1}{m} \sum_{i=1}^{m} Z_i \right|$$

where Z_i is a random variable of mean 0, they are independent.

(To be continued).