Lecture 3: Reconstruction formulae.

This section discusses some methods of reconstruction. The following is the exact reconstruction formula.

**Theorem (Exact Reconstruction formula)**

Let \((\psi_i)_{i=1}^M\) be a frame for \(H^N\), and let \(S\) be the frame operator. Then for every \(x \in H^N\), we have

\[
x = \sum_{i=1}^M <x, \psi_i> S^t \psi_i = \sum_{i=1}^M <x, S^t \psi_i> \psi_i.
\]

\(\square\)

**Proof:** As \(S\) is invertible,

\[
x = S^t S x = S^t \left( \sum_{i=1}^M <x, \psi_i> \psi_i \right) = \sum_{i=1}^M <x, \psi_i> S^t \psi_i,
\]

or

\[
x = S S^t x = \sum_{i=1}^M <S x, \psi_i> \psi_i = \sum_{i=1}^M <x, S^t \psi_i> \psi_i.
\]

\(\square\)
Definition: Let \( \{\psi_i\}_{i=1}^M \) be a frame. The set of vectors given by \( \{S^\dagger \psi_i\}_{i=1}^M \) is called the **canonical dual frame**.

In matrix notation, if \( \Phi \) is the frame matrix, \( S = \Phi \Phi^* = T^* T \).

Thus the canonical dual frame is given by

\[
S^\dagger \Phi = (\Phi \Phi^*)^{-1} \Phi
= (T^* T)^{-1} T^*.
\]

**Proposition:** \( \{S^\dagger \psi_i\}_{i=1}^M \) is a frame with frame bound \( \frac{1}{A_{opt}} \) and \( \frac{1}{B_{opt}} \).

**Proof:**

\[
Q^\dagger S Q = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix}, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N
\]

\[
Q^\dagger S^{-1} Q = \begin{bmatrix} \tilde{\lambda}_1 & & \\ & \ddots & \\ & & \tilde{\lambda}_N \end{bmatrix}
\]

This shows the optimal frame bound are

\[
\frac{1}{B_{opt}} \quad \frac{1}{A_{opt}}.
\]
The following proposition shows the least square property of the canonical frame coefficients.

**Proposition (Least square property of \( \{<x, S^i \phi_i>\}_{i=1}^M \))**

Let \( \{\phi_i\}_{i=1}^M \) be a frame for \( \mathbb{R}^N \). Suppose that \( x = \sum_{i=1}^M c_i \phi_i \), for another set of coefficients \( \{c_i\}_{i=1}^M \). Then

\[
\sum_{i=1}^M |c_i|^2 = \sum_{i=1}^M \left| <x, S^i \phi_i> \right|^2 + \sum_{i=1}^M \left| c_i - <x, S^i \phi_i> \right|^2
\]

In particular,

\[
\sum_{i=1}^M \left| <x, S^i \phi_i> \right|^2 = \inf \left\{ \sum_{i=1}^M |c_i|^2 : x = \sum_{i=1}^M c_i \phi_i \right\}
\]

**P.f:** As \( x = \sum_{i=1}^M c_i \phi_i = \sum_{i=1}^M <x, S^i \phi_i> \phi_i \),

\[
\sum_{i=1}^M (c_i - <x, S^i \phi_i>) \phi_i = 0
\]

Thus \( (c_i - <x, S^i \phi_i>) \phi_i \in \ker \Phi \),

\[
= (\text{range } \Phi^*)^\perp
\]
On the other hand,

\[ \langle x, S^* \phi_i \rangle = \langle S^* x, \phi_i \rangle \]

is in the range of \( S^* \).

As \( C_i \equiv (C_i - \langle x, S^* \phi_i \rangle) + \langle x, S^* \phi_i \rangle \)

is in the kernel of \( S \) or in the range of \( S^* \),

which are orthogonal subspaces, so

\[ \| C_i \|^2 = \| (C_i - \langle x, S^* \phi_i \rangle) \|^2 + \| \langle x, S^* \phi_i \rangle \|^2 \]

(Pythagorean Theorem).

We therefore obtain

\[ \sum_{i=1}^{M} C_i^2 = \sum_{i=1}^{M} | C_i - \langle x, S^* \phi_i \rangle |^2 + \sum_{i=1}^{M} | \langle x, S^* \phi_i \rangle |^2. \]

From the identity, it is clearly true that

\[ \sum_{i=1}^{M} C_i^2 \geq \sum_{i=1}^{M} | \langle x, S^* \phi_i \rangle |^2, \]

which shows the 2nd part. \( \square \)
Remark: ① The proof works for infinite dimension.

② In picture,

\[ x = \sum \xi \varphi_i = \Xi \zeta \]

All \( \zeta \) are in the hyperplane

\[ \hat{x} = \Xi \zeta \]

③ In other problems, there is a need to find \( \hat{x} \) with the most amount of zeros coordinates. This is one of the origin of "compressive sensing".

In the next section, we will see reconstruction can also be performed using another set of frames.

i.e.

\[ x = \sum_{i=1}^{M} \langle x, \psi_i \rangle \psi_i \]

where \( \{ \psi_i \}_{i=1}^{M} \) is called a dual frame.
Frame algorithm:

In practice, $S^{-1}$ may be computational expensive. The following gives an algorithm for reconstruction.

**Theorem (Frame Algorithm).**

Let $(y_{i})_{i=1}^{M}$ be a frame for $\mathcal{H}^{N}$ with frame bounds $A$ and $B$, and frame operator $S$. Given $x \in \mathcal{H}^{N}$, define

$$y_{0} = 0,$$

$$y_{j} = y_{j-1} + \frac{2}{A+B} S(x-y_{j-1})$$

$$= y_{j-1} + \frac{2}{A+B} \left( \sum_{i=1}^{M} \langle x, y_{i} \rangle \langle y_{i}, y_{j-1} \rangle - S y_{j-1} \right)$$

Then $y_{j}$ converges to $x$, with

$$\|x - y_{j}\| \leq \left( \frac{B-A}{B+A} \right)^{2} \|x\|.$$
\[ \begin{align*}
\| x \|_1^2 & - \frac{2A}{A+B} \| x \|_1^2 \\
& = \frac{B-A}{B+A} \| x \|_1^2
\end{align*} \]

Similarly
\[ \langle (I - \frac{2}{A+B}S)x, x \rangle \geq \| x \|_1^2 - \frac{2B}{A+B} \| x \|_1^2 \]
\[ = - \frac{B-A}{B+A} \| x \|_1^2 \]

This implies that
\[ \| I - \frac{2}{A+B}S \| \leq \frac{B-A}{B+A} \]

and
\[ x - y_3 = x - y_{3-1} - \frac{2}{A+B} S(x - y_{3-1}) \]
\[ \Rightarrow \quad (I - \frac{2}{A+B}S)(x - y_{3-1}) \]
\[ x - y_3 = (I - \frac{2}{A+B}S)^3(x - y_0) \]
\[ \| x - y_3 \| \leq \left( \frac{B-A}{B+A} \right)^3 \| x \|_1 \]

This identity shows that \( y_3 \to x \).

The result is proved. \( \square \)
Note that the rate of convergence depends sensitively on the condition number \( \frac{B}{A} \), since we see that

\[
\frac{B - A}{B + A} = \frac{\frac{B}{A} - 1}{\frac{B}{A} + 1}
\]

\( \frac{B}{A} \sim 1 \), \( \frac{D - A}{B + A} \sim 0 \).

If \( \frac{B}{A} = 1 \), the tight frame, \( S = A I_d \)

\[
S^{-1} = \frac{1}{A} I_d
\]

\[
X = \sum_{i=1}^{M} \langle x, \psi_i \rangle \frac{1}{A} \psi_i = \frac{1}{A} \sum_{i=1}^{M} \langle x, \psi_i \rangle \psi_i
\]

There are some other faster frame algorithms.

Characterization of all dual frames

Recall that \((\psi_i)_{i=1}^M\) is a dual frame of the frame \((\phi_i)_{i=1}^M\) if \(\forall x \in \mathcal{H}\).

\[ x = \sum_{i=1}^M \langle x, \phi_i \rangle \psi_i \]

Let \(\Phi = (\phi_1, \ldots, \phi_M)\) and \(\Psi = (\psi_1, \psi_2, \ldots, \psi_M)\) be the corresponding frame matrices.

Then

\[ x = \sum_{i=1}^M \langle x, \phi_i \rangle \psi_i \]

\[ = \overline{\Phi} \begin{bmatrix} \langle x, \phi_1 \rangle \\ \langle x, \phi_2 \rangle \\ \vdots \\ \langle x, \phi_M \rangle \end{bmatrix} \]

\[ = \overline{\Phi} \Phi^* x \]

**Proposition** \(\overline{\Phi}\) is a dual frame of \(\Phi\) if and only if \(\overline{\Phi} \Phi^* = \mathbb{I}\).

\[ \begin{bmatrix} \overline{\Phi}_1 \\ \overline{\Phi}_2 \\ \vdots \\ \overline{\Phi}_M \end{bmatrix} \begin{bmatrix} \Phi_1^* \\ \Phi_2^* \\ \vdots \\ \Phi_M^* \end{bmatrix} = \mathbb{I} \]
Hence \((\mathcal{F}, \mathcal{F}^*)\) is a dual frame pair if \(\mathcal{F}^{\*\*} = \mathcal{F}\) (\(\iff\) \(\mathcal{F}^* \mathcal{F}^* = \mathcal{I}\)).

Expanding it out, we have

\[
\chi = \sum_{i=1}^{M} \langle \chi, \psi_i \rangle \psi_i = \sum_{i=1}^{M} \langle \chi, \psi_i \rangle \psi_i.
\]

\[\text{Eq.}\quad \text{For the canonical dual frame, } (s_i^* \psi_i)_{i=1}^M\]

we can write it as

\[
(s_i^* \psi_i)_{i=1}^M = (\mathcal{F}^* \mathcal{F})^{-1} \mathcal{F}^* \psi_i,
\]

\(i = 1, 2, \ldots, M.\)

Thus, the frame matrix for the canonical dual frame is

\[
(\mathcal{F}^* \mathcal{F})^{-1} \mathcal{F}^* \mathcal{F}, \quad \text{or } s_i^* \mathcal{F}
\]

Of course,

\[
(s_i^* \mathcal{F})^* \mathcal{F}^* = (\mathcal{F}^* \mathcal{F})^{-1} \mathcal{F}^* \mathcal{F}^* = \mathcal{I}.
\]

Now, we want to find all other duals.
This amounts to finding all left inverses of $\Phi^*$. 

$$\overline{\Phi} \Phi^* = I$$

**Theorem (Characterization of all dual frames)**

Let $\overline{\Phi}$ be a dual frame of $\Phi$. Then the following are equivalent:

1. $\overline{\Phi} = \overline{\Phi}_0 + U$ is a dual frame of $\Phi$
2. $U \Phi^* = 0$
3. $\Phi U^* = 0$
4. Rows of $U$ are in the kernel of $\Phi$.

**Proof:** $2 \iff 3 \iff 4$ are standard

1. $\Rightarrow 2$ 
   $$\overline{\Phi} \Phi^* = I$$
   $$\overline{\Phi}_0 \Phi^* + U \Phi^* = I$$
   Hence 
   $$U \Phi^* = I - \Phi_0 \Phi^*$$
   $$= I - I = 0$$

2. $\Rightarrow 1$ 
   $$\overline{\Phi} \Phi^* = (\overline{\Phi}_0 + U) \Phi^*$$
   $$= \overline{\Phi}_0 \Phi^* + U \Phi^* = I.$$
Remark: If \( \Phi_0 = \text{canonical dual frame} \)
\[
\Phi_0 = S^t \Phi
\]
then all dual frames are column vectors of \( S^t \Phi + U \)
where \( U \Phi^* = 0 \)

In other words, the set of all duals forms a \( N(M-N) \)-dimensional subspace of vector space of \( M_{N \times M} \). (Why?)

Ex. Consider the Mercedes-Benz vector
\[
\Phi = \begin{bmatrix}
0 & \sqrt{3}/2 & -\sqrt{3}/2 \\
1 & -\frac{1}{2} & -\frac{1}{2}
\end{bmatrix}
\]
To find the dual frames of \( \Phi \), we need the canonical dual \( S^t \Phi = \frac{2}{3} \Phi \)

We now find \( U \).

As \( \Phi U^* = 0 \), we solve the

\[
\ker \Phi : \begin{bmatrix}
0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\
1 & -\frac{1}{2} & -\frac{1}{2}
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
\[
\begin{align*}
\begin{cases}
y - z = 0 \\
x - \frac{1}{2} y - \frac{1}{2} z = 0
\end{cases}
\end{align*}
\]

Thus \(x = y = z\) and

\[
\ker \Xi = \left\{ \begin{bmatrix} x \\ x \\ x \end{bmatrix} : x \in \mathbb{R} \right\}
\]

Here

\[
\mathcal{U} = \begin{bmatrix} x & y \\ x & y \\ x & y \end{bmatrix}
\]

and we have all dual frames are

\[
\frac{2}{3} \Xi^* + \begin{bmatrix} x & x & x \\ y & y & y \end{bmatrix}
\]

or

\[
\frac{2}{3} \begin{bmatrix} 0 & 2^{3/2} & -2^{3/2} \\ 1 & -1 & -1 \end{bmatrix} + \begin{bmatrix} x & x & x \\ y & y & y \end{bmatrix}.
\]

**Theorem (Li's formula)**

Let \(\Xi\) be a frame and \(\Xi_0\) be a dual frame. Then the set of all dual frames are given by

\[
\Xi = \Xi_0 + E \left( I - \Xi^* \Xi_0 \right)
\]

where \(E\) is any matrix of the right size.
pf: We just need to show $\Phi \Phi^* = I$.

Clearly

$$\left[ \Phi_0 + E\left(\Gamma - \Phi^* \Phi \right) \right] \Phi^*$$

$$= I + E \left( \Phi^* - \Phi^* \right) = I$$

so the formula gives us a dual frame.

Conversely, if $\Phi$ is a dual frame, we let $E = \Phi$, then

$$\Phi_0 + E\left(\Gamma - \Phi^* \Phi \right)$$

$$= \Phi_0 + \left( \Phi^* - \Phi^* \right) = \Phi.$$  

This formula gives all the dual frames $\Box$

Remark: These formulae will form the basis of finding the optimal dual frames for a specific optimization problem.

Remark: All formulae and theorems we discussed works perfectly in infinite dimensional space.