Lecture 3 : Reconstruction formulae.

This section discuss some methods of reconstruction. The following is the exact reconstruction formula.

Theorem (Exact Reconstruction formula)

Let $(\varphi_i)_{i=1}^{M}$ be a frame for $\mathcal{H}^N$, and let S be the frame operator. Then for every $x \in \mathcal{H}^N$, we have

$$x = \sum_{i=1}^{M} \langle x, \varphi_i \rangle S^{-1}\varphi_i = \sum_{i=1}^{M} \langle x, S^{-1}\varphi_i \rangle \varphi_i$$

Pf: As S is invertible,

$$\begin{aligned} x &= S^{-1} S x \\ &= S^{-1}\left(\sum_{i=1}^{M} \langle x, \varphi_i \rangle \varphi_i \right) \\ &= \sum_{i=1}^{M} \langle x, \varphi_i \rangle S^{-1}\varphi_i \end{aligned}$$

or

$$\begin{aligned} x &= S S^{-1} x \\ &= \sum_{i=1}^{M} \langle S^{-1}x, \varphi_i \rangle \varphi_i = \sum_{i=1}^{M} \langle x, S^{-1}\varphi_i \rangle \varphi_i \end{aligned}$$

$\square$

Definition: Let $(\varphi_i)_{i=1}^M$ be a frame, The set of vectors given by $(S^{-1}\varphi_i)_{i=1}^M$ is called the <u>canonical dual frame</u>.

In Matrix Notation, If Φ is the frame matrix, $S = \Phi\Phi^* = T^*T$.

Thus the canonical dual frame is given by

$$S^{-1}\Phi = (\Phi\Phi^*)^{-1}\Phi$$
$$= (T^*T)^{-1}T^*.$$

Proposition: $\{S^{-1}\varphi_i\}_{i=1}^M$ is a frame with frame bound $\frac{1}{A_{opt}}$ and $\frac{1}{B_{opt}}$

p.f.:
$$Q^T S Q = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix}, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$$

$$Q^T S^{-1} Q = \begin{bmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_N^{-1} \end{bmatrix}$$

This shows the optimal frame bound are

$$\frac{1}{B_{opt}} \qquad \frac{1}{A_{opt}}.$$

The following Proposition shows the least square property of the canonical frame coefficients.

<u>Proposition</u> (Least square property of $\{\langle x, S^{-1}\varphi_i\rangle\}_{i=1}^{N}$)

Let $\{\varphi_i\}_{i=1}^{M}$ be a frame for $\mathcal{H}^N$. Suppose that $x = \sum_{i=1}^{M} c_i \varphi_i$, for another set of coefficients $\{c_i\}_{i=1}^{M}$. Then

$$\sum_{i=1}^{M} |c_i|^2 = \sum_{i=1}^{M} |\langle x, S^{-1}\varphi_i\rangle|^2 + \sum_{i=1}^{M} \left| c_i - \langle x, S^{-1}\varphi_i\rangle \right|^2$$

In Particular,

$$\sum_{i=1}^{M} |\langle x, S^{-1}\varphi_i\rangle|^2 = \inf\left\{ \sum_{i=1}^{M} |c_i|^2 : x = \sum_{i=1}^{M} c_i \varphi_i \right\}$$

<u>Pf</u>: As $x = \sum_{i=1}^{M} c_i \varphi_i = \sum_{i=1}^{M} \langle x, S^{-1}\varphi_i\rangle \varphi_i$

$$\therefore \sum_{i=1}^{M} \left(c_i - \langle x, S^{-1}\varphi_i\rangle\right)\varphi_i = 0$$

Thus $\left(c_i - \langle x, S^{-1}\varphi_i\rangle\right)_{i=1}^{M} \in \ker \Phi$.

$$= \left(\text{range } \Phi^*\right)^{\perp}$$

On the other hand

$$\langle x, S^{-1}\varphi_i \rangle = \langle S^{-1}x, \varphi_i \rangle$$

$$\in \text{range } \Phi^* .$$

As $$(c_i) = \left(c_i - \langle x, S^{-1}\varphi_i \rangle \right) + \left(\langle x, S^{-1}\varphi_i \rangle \right)$$

$$\in \ker \Phi \oplus \text{range } \Phi^* ,$$

which are orthogonal subspaces, so.

$$\| (c_i) \|^2 = \| \left(c_i - \langle x, S^{-1}\varphi_i \rangle \right) \|^2 + \| \langle x, S^{-1}\varphi_i \rangle \|^2$$

(Pyth. Theorem)

We therefore obtain

$$\sum_{i=1}^{M} |c_i|^2 = \sum_{i=1}^{M} | c_i - \langle x, S^{-1}\varphi_i \rangle |^2 + \sum_{i=1}^{M} | \langle x, S^{-1}\varphi_i \rangle |^2 .$$

From the identity, it is clearly true that

$$\sum_{i=1}^{M} |c_i|^2 \geq \sum_{i=1}^{M} | \langle x, S^{-1}\varphi_i \rangle |^2 ,$$

which shows the 2nd part. $\square$

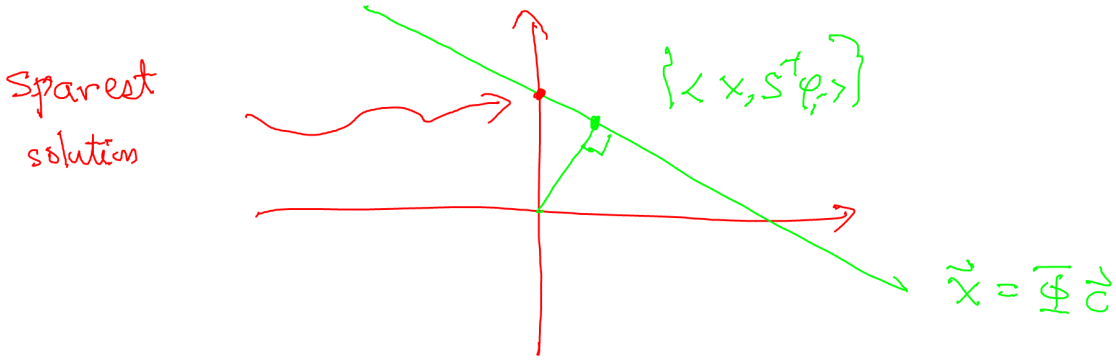
Remark: ① The proof works for infinite dimension.

② In picture,

$$x = \sum c_i \varphi_i = \Phi \vec{c}$$

All $\vec{c}$ are in the hyperplane

$$\vec{x} = \Phi \vec{c}$$



③ In other problems, there is a need to find $\vec{c}$ with the most amount of zeros coordinates. This is one of the origin of "compressive sensing".

In the next section, we will see reconstruction can also be performed using another set of frames.

i.e. $$x = \sum_{i=1}^{M} \langle x, \varphi_i \rangle \psi_i$$

where $\{\psi_i\}_{i=1}^{M}$ is called a <u>dual frame</u>.

Frame algorithm:

In practice, S^{-1} may be computational expensive. The following gives an algorithm for reconstruction.

Theorem (Frame algorithm).

Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathcal{H}^N$ with frame bounds A and B, and frame operator S. Given $x \in \mathcal{H}^N$, define

$$y_0 = 0 \qquad y_j = y_{j-1} + \frac{2}{A+B} S(x - y_{j-1})$$

$$= y_{j-1} + \frac{2}{A+B}\left(\sum_{i=1}^{M} \langle x, \varphi_i \rangle \varphi_i - S y_{j-1} \right)$$

Then y_j converges to x. with

$$\|x - y_j\| \leq \left(\frac{B-A}{B+A} \right)^j \|x\|.$$

Pf.: Let us compute

$$\left\langle \left(Id - \frac{2}{A+B} S \right) x, x \right\rangle = \|x\|^2 - \frac{2}{A+B} \langle Sx, x \rangle$$

$$= \|x\|^2 - \frac{2}{A+B} \sum_{i=1}^{M} |\langle x, \varphi_i \rangle|^2$$

$$\leq \|x\|^2 - \frac{2A}{A+B}\|x\|^2$$

$$= \frac{B-A}{B+A}\|x\|^2$$

Similarly $\left\langle \left(Id - \frac{2}{A+B}S\right)x, x\right\rangle \geq \|x\|^2 - \frac{2B}{A+B}\|x\|^2$

$$= -\frac{B-A}{B+A}\|x\|^2$$

This implies that

$$\left\| Id - \frac{2}{A+B}S \right\| \leq \frac{B-A}{B+A}.$$

and

$$x - y_j = x - y_{j-1} - \frac{2}{A+B}S(x - y_{j-1})$$

$$\Rightarrow \quad = \left(Id - \frac{2}{A+B}S\right)(x - y_{j-1})$$

$$x - y_j = \left(Id - \frac{2}{A+B}S\right)^j (x - y_0)$$

$$\|x - y_j\| \leq \left(\frac{B-A}{B+A}\right)^j \|x\|.$$

This identity shows that $y_j \to x$.

The result is proved. $\square$

Note that the rate of convergence depends sensitively on the condition number. B/A, since we see that

$$\frac{B-A}{B+A} = \frac{\frac{B}{A}-1}{B/A+1}$$

$$B/A \sim 1, \quad \frac{B-A}{B+A} \sim 0 \; !$$

If $\frac{B}{A} = 1$, the tight frame. $S = A\,Id$

$$S^{-1} = \frac{1}{A} Id$$

$$x = \sum_{i=1}^{M} \langle x, \varphi_i \rangle \frac{1}{A} \varphi_i = \frac{1}{A} \sum_{i=1}^{M} \langle x, \varphi_i \rangle \varphi_i .$$

There are some other faster frame algorithm,

See Grochenig, Acceleration of the frame algorithms, IEEE Trans. Signal Process. 41 3331 – 3340 (1993)

Characterization of all dual frames

Recall that $(\psi_i)_{i=1}^M$ is a dual frame of the frame $(\varphi_i)_{i=1}^M$ if $\forall x \in \mathcal{H}^N$.

$$x = \sum_{i=1}^{M} \langle x, \varphi_i \rangle \psi_i$$

Let $\Phi = \begin{pmatrix} | & & | \\ \varphi_1 & \cdots & \varphi_M \\ | & & | \end{pmatrix}$ and $\Psi = \begin{pmatrix} | & | & & | \\ \psi_1 & \psi_2 & \cdots & \psi_M \\ | & | & & | \end{pmatrix}$

be the corresponding frame matrix.

Then
$$\begin{aligned} x &= \sum_{i=1}^{M} \langle x, \varphi_i \rangle \psi_i \\ &= \Psi \begin{bmatrix} \langle x, \varphi_1 \rangle \\ \langle x, \varphi_2 \rangle \\ \vdots \\ \langle x, \varphi_M \rangle \end{bmatrix} \\ &= \Psi \Phi^* x \end{aligned}$$

<u>Proposition</u> Ψ is a dual frame of Φ if and only if $\Psi \Phi^* = I$

$$\Updownarrow$$

$$\Phi \Psi^* = I.$$

Hence (Φ, Φ) is a dual frame pair

if $\Phi \Phi^* = I$ $\left(\Leftrightarrow \Phi^* \Phi = I \right)$

Expanding it out, we have

$$x = \sum_{i=1}^{M} \langle x, \varphi_i \rangle \psi_i = \sum_{i=1}^{M} \langle x, \psi_i \rangle \varphi_i .$$

<u>E.g.</u> For the canonical dual frame $(S^{-1}\varphi_i)_{i=1}^{M}$

we can write it as

$$(\Phi \Phi^*)^{-1} \varphi_i \quad , \quad i = 1, 2, \cdots, M.$$

Thus, the frame matrix for the canonical dual frame

is $(\Phi \Phi^*)^{-1} \Phi$. or $S^{-1} \Phi$

Of course,

$$(S^{-1} \Phi) \Phi^* = (\Phi \Phi^*)^{-1} \Phi \Phi^* = I.$$

Now, we want to find all other duals.

This amounts to finding all <u>left inverses</u> of Φ^*.

$$\Psi \Phi^* = I$$

<u>Theorem</u> (Characterization of all dual frames)

Let Ψ_0 be a dual frame of Φ. Then the following are equivalent

① $\Psi = \Psi_0 + U$ is a dual frame of Φ

② $U \Phi^* = 0$

③ $\Phi U^* = 0$

④ Rows of U are in the kernel of Φ.

<u>Proof</u>: ② $\Leftrightarrow$ ③ $\Leftrightarrow$ ④ are standard

① $\Rightarrow$ ② $\quad \Psi \Phi^* = I$

$$\Psi_0 \Phi^* + U \Phi^* = I$$

Hence $\quad U \Phi^* = I - \Psi_0 \Phi^*$

$$= I - I = 0$$

② $\Rightarrow$ ① $\quad \Psi \Phi^* = (\Psi_0 + U) \Phi^*$

$$= \Psi_0 \Phi^* + U \Phi^* = I.$$

<u>Remark</u>: If Φ_0 = canonical dual frame

$$= S^{-1}\Phi$$

then all dual frames are column vectors of $S^{-1}\Phi + U$

where $U\Phi^* = \underline{0}$

In other words, the set of all duals forms a $N(M-N)$ - dimensional subspace of vector space of $M_{N\times M}$. (Why?)

Eg. Consider the Mercedes - Benz vector

$$\Phi = \begin{bmatrix} 0 & \sqrt{3}/2 & -\sqrt{3}/2 \\ 1 & -1/2 & -1/2 \end{bmatrix}$$

To find the dual frames of Φ, we need the canonical dual $S^{-1}\Phi = \frac{2}{3}\Phi$

We now find U.

As $\Phi U^* = \underline{0}$, we solve the

$\ker \Phi$:

$$\begin{bmatrix} 0 & \sqrt{3}/2 & -\sqrt{3}/2 \\ 1 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} y - z = 0 \\ x - \frac{1}{2}y - \frac{1}{2}z = 0 \end{cases}$$

Thus $x = y = z$ and

$$\ker \Phi = \left\{ \begin{bmatrix} x \\ x \\ x \end{bmatrix} : x \in \mathbb{R} \right\}$$

Hence $U^* = \begin{bmatrix} x & y \\ x & y \\ x & y \end{bmatrix}$ and we have all dual frames are

$$\frac{2}{3}\Phi + \begin{bmatrix} x & x & x \\ y & y & y \end{bmatrix}$$

or $\frac{2}{3}\begin{bmatrix} 0 & \sqrt{3}/2 & -\sqrt{3}/2 \\ 1 & -1/2 & -1/2 \end{bmatrix} + \begin{bmatrix} x & x & x \\ y & y & y \end{bmatrix}$ #.

<u>Theorem</u> (Li's formula)

Let Φ be a frame and Φ_0 be a dual frame. Then the set of all dual frames are given by

$$\Phi = \Phi_0 + E(I - \Phi^* \Phi_0)$$

where E is any matrix of the right size.

Pf: We just need to show $\Phi \Phi^* = I$.

Clearly $\left[\Phi_0 + E(I - \Phi^* \Phi_0)\right] \Phi^*$

$$= I + E(\Phi^* - \Phi^*) = I.$$

So the formula gives us a dual frame.

Conversely, if Φ is a dual frame, we let

$E = \Phi$, then

$$\Phi_0 + \Phi(I - \Phi^* \Phi_0)$$

$$= \Phi_0 + (\Phi - \Phi_0) = \Phi.$$

This formula gives all the dual frames $\square$

<u>Remark</u>: These formulae will form the basis of finding the optimal dual frames for a specific optimization problem.

<u>Remark</u>: All formulae and theorem we discussed works perfectly in infinite dimensional space.