Lecture 6: Frames, Projection and other models.

An operator P is called an orthogonal projection if $P^2 = P$ and $P^* = P$

E.g. If $\vec{\varphi}$ is a unit vector. The projection to the subspace $\text{span}\{\vec{\varphi}\}$ is given by

$$Px = \langle x, \vec{\varphi} \rangle \vec{\varphi} = \varphi \varphi^* x$$

or $P = \vec{\varphi} \vec{\varphi}^*$.

<u>Theorem</u>: Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathcal{H}^N$, with frame bound A and B. Let P be an orthogonal projection of $\mathcal{H}^N$ onto a subspace $W \subseteq \mathcal{H}^N$.

Then $\{P\varphi_i\}_{i=1}^M$ is a frame of W.

Conversely, if $(\varphi_i)_{i=1}^M$ is a frame for a subspace $W \subseteq \mathcal{H}^N$. Then the orthogonal projection of W is given by

$$Px = \sum_{i=1}^M \langle x, S^{-1}\varphi_i \rangle \varphi_i .$$

Pf.:

$$A\|x\|^2 \le \sum_{i=1}^{M} |\langle x, \varphi_i \rangle|^2 \le B\|x\|^2.$$

Take $x \in W$, and put Px into the definition

$$A\|Px\|^2 \le \sum_{i=1}^{M} |\langle Px, \varphi_i \rangle|^2 \le B\|Px\|^2$$

But $\|Px\|^2 = \langle x, P^2 x \rangle = \langle x, Px \rangle = \langle x, x \rangle = \|x\|^2$.

we are done

For the second part, we note that if $x \in W$, then $x = Px$

Thus
$$Px = x = \sum_{i=1}^{M} \langle x, S^{-1}\varphi_i \rangle \varphi_i$$
since $(\varphi_i)_{i=1}^{M}$ is a frame for W.

If $x \in W^{\perp}$, then $\langle x, S^{-1}\varphi_i \rangle = 0$

since $S : W \to W$ is bijective, so is S^{-1}. Thus $Px = 0$.

In general, $x = x'' + x^{\perp} \in W \oplus W^{\perp}$

$$Px = Px'' + Px^{\perp} = Px''$$
$$= \sum_{i=1}^{M} \langle x, S^{-1}\varphi_i \rangle \varphi_i$$

$\#$

<u>Corollary</u>: Let $(e_i)_{i=1}^N$ be an orthonormal basis for $\mathcal{H}^N$. Let P be an orthogonal projection for $\mathcal{H}^N$ onto W. Then $(Pe_i)_{i=1}^N$ is a Parseval frame.

Conversely, Parseval frame $(\varphi_i)_{i=1}^M$ is a compression of an orthonormal basis. i.e. there exists orthogonal projection P on $\mathcal{H}^M$ to $\mathcal{H}^N$ and orthonormal basis $(e_i)_{i=1}^M$ such that

$$Pe_i = \varphi_i .$$

(Naimark Theorem)

The note has given a coordinate independent proof. However, we can prove it by coordinates, and use Gram-Schmidt process.

Frames operator from projection point of view

Let $\Phi = (\varphi_i)_{i=1}^{M}$ be a frame on $\mathcal{H}^N$. Assume further $\|\varphi_i\| = 1$. (Unit Norm frame). Then the frame operator is given by

$$S = \Phi \Phi^*.$$
$$= \sum_{i=1}^{M} \varphi_i \varphi_i^*.$$

where $\varphi_i \varphi_i^*$ is the orthogonal projection operator onto span $\{\varphi_i\}$.

Note that $P_i x = \varphi_i \varphi_i^* x$

The frame inequality can be thought as

$$A \|x\|^2 \leq \sum_{i=1}^{M} \|P_i x\|^2 \leq B \|x\|^2.$$

This allows us to consider different model of frame reconstruction.

Fusion frame

Let $(W_i)_{i\in I}$ be a collection of subspaces in $\mathcal{H}^N$.

Let $v_i > 0$ be a set of weights. Then,

we say that $(W_i, v_i)_{i\in I}$ is a fusion frame if $\exists\ A > 0,\ A \le B$

$$A\|x\|^2 \le \sum_{i\in I} v_i \|P_{W_i} x\|^2 \le B\|x\|^2 \quad \forall x \in \mathcal{H}^N.$$

where P_{W_i} is the orthogonal projection onto W_i.

Remark: when $W_i = \operatorname{span}\{\varphi_i\}$, $v_i = \|\varphi_i\|^2$. It reduces to the frame definition.

Fusion frame concerns reconstruction of signals by the data $\{P_{W_i} x\}_{i\in I}$

Phase retrieval

Phase retrieval concerns recovering vector $x \in \mathbb{H}^N$ from $\{ |\langle x, \varphi_i \rangle| \}_{i=1}^{M}$

<u>Definition</u>: We say that vectors $\{\varphi_i\}_{i=1}^{M}$ does <u>phase retrieval</u> if

$$|\langle x, \varphi_i \rangle| = |\langle y, \varphi_i \rangle| \quad \forall\, i = 1, \dots M$$

then $\quad x = y\, e^{i\varphi}$. $\quad$ (on $\mathbb{C}^N$)

or $\quad x = \pm y \quad$ (on $\mathbb{R}^N$).

Phase retrieval by projection

We say that the $\{P_{W_i}\}$ operators do phase retrieval if

$$|P_{W_i} x| = |P_{W_i} y|$$

implies $\quad x = y \quad$ up to a phase.

Sigma - Delta quantization (Σ-Δ)

In signal transmission, the transmitted signal are $\{\langle x, \varphi_i \rangle\}_{i=1}^{M}$ and then reconstruct it using $\sum_{i=1}^{M} \langle x, \varphi_i \rangle \psi_i$.

$\{\langle x, \varphi_i \rangle\}$ are in $\mathbb{C}^M$, they are not discrete or digitalized at all.

We do a <u>quantization</u> in the channel.

$$Q : \{\langle x, \varphi_i \rangle\} \longmapsto \{q_i\}$$

where $\{q_i\} \in \mathcal{A}^M$ and

$\mathcal{A}$ is a set of finite alphabet.

Then we construct it by

$$\tilde{x} = \sum_{i=1}^{M} q_i e_i$$

Qu: Estimate $\|x - \tilde{x}\|$. ?

In general, what we are doing are

$$\boxed{x} \xrightarrow{\Phi \text{ : frame}} \boxed{\Phi^* x}$$

$$\boxed{\Phi^* x} \xrightarrow{\text{Some maps } T} \boxed{T(\Phi^* x)}$$

$$\boxed{T(\Phi^* x)} \xrightarrow{\text{reconstruct}} \boxed{\tilde{x}}$$

Find $\|x - \tilde{x}\|$?

Phase retrieval : $\Phi^* x \longmapsto \{ |\langle x, \varphi_i \rangle| \}_{i=1}^{M}$

$\Sigma\Delta$ - Quantization : $\Phi^* x \longmapsto \{ q_n \}_{i=1}^{M}$

Erasure Map : $\Phi^* x \longmapsto \{ D \Phi^* x \}$

$$D = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \ddots \\ & & & & 1 \end{bmatrix}$$

1 : For non-erasured position

0 : For erasured position.

<u>References</u> : Google the title with the authors name, you will find their papers.

<u>Fusion Frames</u> : S. Li, Casazza and Kutyinok
Y. Wang, D. Mixon, etc

<u>Phase retrieval</u> : Casazza, J. Cahill.
J. Treiman.

<u>$\Sigma\Delta$ - Quantization</u>: Benedetto, Powell, Yilmaz
and many others.

<u>Erasure</u> : D. Han, D. Mixon, M. Fickus.

<u>Tight frame Construction</u> : D. Mixon, M. Fickus.