Lecture 6: Frames, Projection and other models.

An operator $P$ is called an orthogonal projection if $P^2 = P$ and $P^* = P$.

E.g., if $\phi$ is a unit vector, the projection to the subspace span $\{\phi\}$ is given by

$$Px = \langle x, \phi \rangle \phi = \phi \phi^* x$$

or $P = \phi \phi^*$.

Theorem: Let $(\phi_i)_{i=1}^M$ be a frame for $\mathbb{C}^N$, with frame bound $A$ and $B$. Let $P$ be an orthogonal projection of $\mathbb{C}^N$ onto a subspace $W \subseteq \mathbb{C}^N$. Then $\{P\phi_i\}_{i=1}^M$ is a frame of $W$.

Conversely, if $(\phi_i)_{i=1}^M$ is a frame for a subspace $W \subseteq \mathbb{C}^N$, then the orthogonal projection of $W$ is given by

$$Px = \sum_{i=1}^M \langle x, S_i \phi_i \rangle \phi_i.$$
For the first part, we note that if \( x \in W \), then \( x = P_x \).

Thus \( P_x = x = \sum_{i=1}^{M} \langle x, \varphi_i^\perp \rangle \varphi_i^\perp \).

Since \((\varphi_i^\perp)_{i=1}^{M}\) is a frame for \( W^\perp \).

If \( x \in W^\perp \), then \( \langle x, S^* \varphi_i \rangle = 0 \).

Since \( S : W \rightarrow W \) is bijective, so is \( S^* \). Thus \( P_x = 0 \).

In general, \( x = x^\perp + x^\parallel \in W \oplus W^\perp \).

\[
P_x = P_x^\perp + P_x^\parallel = P_x^\parallel = \sum_{i=1}^{M} \langle x, \varphi_i \rangle \varphi_i
\]
Corollary: Let \((e_i)_{i=1}^{N}\) be an orthonormal basis for \(H^N\). Let \(P\) be an orthogonal projection for \(H^N\) onto \(W\). Then \((Pe_i)_{i=1}^{N}\) is a Parseval frame.

Conversely, Parseval frame \((e_i)_{i=1}^{M}\) is a compression of an orthonormal basis, i.e., there exists orthogonal projection \(P\) on \(H^M\) to \(H^N\) and orthonormal basis \((e_i)_{i=1}^{M}\) such that

\[Pe_i = e_i.\]

(Naimark Theorem)

The note has given a coordinate independent proof. However, we can prove it by coordinates and use Gram-Schmidt process.
Frames operator from projection point of view

Let \( \mathbf{F} = (\psi_i)^M_{i=1} \) be a frame on \( \mathbb{H}^N \). Assume further \( \|\psi_i\| = 1 \). (Unit Norm frame). Then the frame operator is given by

\[
S = \mathbf{F} \mathbf{F}^* = \sum_{i=1}^{M} \psi_i \psi_i^*.
\]

where \( \psi_i \psi_i^* \) is the orthogonal projection operator onto \( \text{span } \psi_i \).

Note that \( P_i x = \psi_i \psi_i^* x \)

The frame inequality can be thought as

\[
A \|x\|^2 \leq \sum_{i=1}^{M} \|P_i x\|^2 \leq B \|x\|^2.
\]

This allows us to consider different model of frame reconstruction.
Fusion frame

Let \((W_i)_{i \in I}\) be a collection of subspaces in \(\mathbb{H}^N\).

Let \(\nu_i > 0\) be a set of weights. Then,
we say that \((W_i, \nu_i)_{i \in I}\) is a **fusion frame** if

\[
\exists \ A > 0, \ A \in \mathbb{R}^+ \quad \forall x \in \mathbb{H}^N
\]

\[
A \|x\|^2 \leq \sum_{i \in I} \nu_i \|P_{W_i}x\|^2 \leq B \|x\|^2
\]

where \(P_{W_i}\) is the orthogonal projection onto \(W_i\).

Remark: when \(W_i = \text{span} \{v_i\}\), \(\nu_i = \|v_i\|^2\), \(x \in \mathbb{H}^N\), it reduces to the frame definition.

Fusion frame concerns reconstruction of signals
by the data \(\{P_{W_i}x\}_{i \in I}\)
Phase retrieval

Phase retrieval concerns recovering vector $x \in \mathbb{C}^N$ from $\{\langle x, \varphi_i \rangle \}_{i=1}^M$.

**Definition:** We say that vectors $\{\varphi_i\}_{i=1}^M$ does phase retrieval if

$$|\langle x, \varphi_i \rangle| = |\langle y, \varphi_i \rangle| \quad \forall i = 1, \ldots, M$$

then $x = y e^{i\phi},$ (on $\mathbb{C}^N$)

or $x = \pm y$ (on $\mathbb{R}^N$).

Phase retrieval by projection

We say that the $\{P_{\varphi_i}\}$ operators do phase retrieval if

$$P_{\varphi_i} x = P_{\varphi_i} y$$

implies $x = y$ up to a phase.
Sigma-Delta quantization ($\Sigma-\Delta$)

In signal transmission, the transmitted signal are \( \{ \langle x, \psi_i \rangle \} \) and then reconstruct it using \( \sum_{i=1}^{M} x, \psi_i, \psi_i \).

\( \{ \langle x, \psi_i \rangle \} \) are in \( \mathbb{C}^M \), they are not discrete or digitalized at all.

We do a quantization in the channel.

\[
Q : \{ \langle x, \psi_i \rangle \} \rightarrow \{ q_i \}
\]

where \( \{ q_i \} \in \mathbb{A}^M \) and \( \mathbb{A} \) is a set of finite alphabet.

Then we construct it by

\[
\tilde{x} = \sum_{i=1}^{M} g_i e_i
\]

Q: Estimate \( \| x - \tilde{x} \|_2 \)?
In general, what we are doing is:

\[ \Phi : \mathcal{X} \rightarrow \Phi \mathcal{X} \]

Some maps \( T \)

\[ T(\Phi \mathcal{X}) \rightarrow \hat{\mathcal{X}} \]

Find \( \| x - \hat{x} \| \) ?

Phase retrieval:
\[ \Phi \mathcal{X} \mapsto \{ |\langle x, y \rangle | \}_{n}^{M} \]

ΣΔ - Quantization:
\[ \Phi \mathcal{X} \mapsto \{ \beta_{n} \}_{n}^{M} \]

Erasure Map:
\[ \Phi \mathcal{X} \mapsto \{ \mathcal{D} \Phi \mathcal{X} \} \]

\[ \mathcal{D} = \begin{bmatrix} \vdots & \cdots & \cdots \end{bmatrix} \]

1: For non-erased position
0: For erased position.
References: Google the title with the authors name, you will find their papers.

Fusion Frames: S. Li, Casazza and Kutyniok Y. Wang, D. Mixon, etc

Phase retrieval: Casazza, J. Cahill, J. Treiman.

$f$-Quantization: Benedetto, Powell, Yılmaz and many others.

Erasure: D. Han, D. Mixon, M. Fickus.