Kadison-Singer problem in the context of frames

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The original Kadison-Singer conjecture by R. Kadison and I. Singer in 1959.

Conjecture

Does every pure state on the (abelian) von Neumann algebra $\mathbb{D}$ of bounded diagonal operators on ℓ^2 have a unique extension to a pure state on $B(\ell^2)$, the von Neumann algebra of all bounded operators on ℓ^2?

Kadison-Singer Problems

It was proved to be equivalent to many open conjectures

1. Andersons paving conjectures
2. Weavers discrepancy conjectures
3. Bourgain-Tzafriri Conjecture
4. Feichtinger Conjecture

Conjecture (Fetchtinger Conjecture)

Every frame $(\varphi_i)_{i=1}^{\infty}$ on a Hilbert space which are norm-bounded below (i.e. $\inf_i \|\varphi_i\| > 0$) must be a finite union of Riesz sequences.

Kadison-Singer Problems

In 2013, A. Marcus, D. Spielman and N. Srivastava, a group of mathematical computer scientists solved the conjecture. He was considering problems on Ramanujan graphs.

Definition

we say that a d-regular graph is Ramanujan if all of the non-trivial eigenvalues of its adjacent matrices lie between $-2\sqrt{d-1}, 2\sqrt{d-1}$.

They constructed bipartite Ramanujan Graphs of all degrees. At the same time, their technique solved the Kadison-Singer Conjecture.

Kadison-Singer Problems

They proved

Theorem

Let r be a positive integer and let $\varphi_1, ..., \varphi_M \in \mathbb{C}^N$ such that

$$\sum_{i=1}^{M} \varphi_i \varphi^* = Id$$

and $\|\varphi_i\| \leq \delta$ for all i. Then there exists a partition $S_1, ..., S_r$ of $\{1, ..., M\}$ such that

$$\|\sum_{i \in S_j} \varphi_i \varphi^*\| \leq \left(\frac{1}{\sqrt{r}} + \sqrt{\delta}\right)^2.$$

1. This theorem implies the Weaver Discrepancy conjecture and hence solved the Kadison-Singer.

Kadison-Singer Problems

The theorem means that we can partition a Parseval frame into r subsets and each one frame has frame bound almost $1/r$.

$$\Phi = \left(\begin{array}{c|c|c} S_1 & \cdots & S_r \end{array} \right)$$

Main Tools

Main Tools

Main Tools

Definition

We say that $g(x) = \alpha_0 \prod_{i=1}^{n-1}(x - \alpha_i)$ *interlaces the polynomial* $f(x) = \beta_0 \prod_{j=1}^{n}(x - \beta_j)$ *if their roots satisfy*

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \ldots \leq \alpha_{n-1} \leq \beta_n.$$

We say that $f_1, \ldots, f_m$ *have a common interlacing* g *if* g *interlaces all* f_i.

Main Tools

The main lemma used

Lemma

Let $f_1, ..., f_k$ be polynomials of the same degree that are real-rooted and have positive leading coefficients. Define

$$f_\emptyset = f_1 + ... + f_k.$$

If $f_1, ..., f_k$ have a common interlacing, then there exists an i so that the largest root of f_i is at most the largest root of $f_\emptyset$.

Main Tools

Interlacing happens in self-adjoint matrices. The following theorem is a classical theorem (Cauchy Interlacing theorem), although it is not enough to solve our problem.

Theorem

Let H be a $n \times n$ self-adjoint matrix and

$$H = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

with A is $m \times m$. Then their eigenvalues

$$\lambda_{k+n-m}(H) \leq \lambda_k(A)\lambda\lambda_k(H), \ k = 1, 2, ..., m.$$

If $m = n - 1$, we have

$$\lambda_n(H) \leq \lambda_{n-1}(A) \leq ... \leq \lambda_1(A) \leq \lambda_1(H).$$

Main Tools

Probabilistic formulation

Theorem

Let $\epsilon > 0$ and let $v_1, ..., v_M \in \mathbb{C}^N$ be independent random vectors with finite support such that

$$\mathbb{E}\left(\sum_{i=1}^{M} v_i v^*\right) = Id$$

and $\mathbb{E}(\|v_i\|^2) \leq \epsilon$ for all i. Then

$$\mathbb{P}\left(\|\sum_{i=1}^{M} v_i v_i^*\| \leq (1+\sqrt{\epsilon})^2\right) > 0.$$

Main Tools

Let $\chi(A)(x) = \det(xI - A)$. Their main technical work are:

1. $\mathbb{E}_{v_1,\dots,v_m}\left(\chi(\sum_{i=1}^m v_i v_i^*)(x)\right) = \sum_{j=1}^r p_j \mathbb{E}_{v_2,\dots,v_m}\left(\chi(w_{j,1}w_{j,1}^* + \sum_{i=2}^m v_i v_i^*)(x)\right)$ where $w_{1,1},\dots,w_{r,1}$ are vectors attained by v_1 with probability $p_1,\dots,p_r$. They showed that the polynomials in the latter sum forms a interlacing.
2. This idea penetrates and produces a tree of interlacing polynomials.
3. Then establish that the largest root of $\mathbb{E}_{v_1,\dots,v_m}\left(\chi(\sum_{i=1}^m v_i v_i^*)(x)\right)$ is less than or equal to $(1+\sqrt{\epsilon})^2$.
4. Invoking the lemma, prove that one of the realization proves our results.

Main Tools

The lemma is an undergraduate level proof. Suppose that α_{n-1} is a largest root of the intelacing polynomial. Then $f_i(\alpha_{n-1}) \leq 0$. Hence, $f_\emptyset(\alpha_{n-1}) \leq 0$. On the other hand, positive leading coefficients tell us that $f > 0$ at positive infinity. Hence, the largest $\beta_n \geq \alpha_{n-1}$, $f_\emptyset(\beta_n) = 0$. But

$$0 = f_\emptyset(\beta_n) = f_1(\beta_n) + ... + f_k\beta_n$$

There exists one i such that $f_i(\beta_n) \geq 0$. That means the largest root of f_i occurs before β_n, this completes the proof.

Main Tools

Probability Model

Define the random vector

$$v_i = \begin{pmatrix} \varphi_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \varphi_i \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \varphi_i \end{pmatrix},$$

on $\mathbb{C}^{rd}$ with each probability $1/r$. Then

$$\mathbb{E}(\sum_{i=1}^{m} v_i v_i^*) = Id.$$

$$\mathbb{E}(\|v_i\|^2) =\leq \epsilon.$$

Open problem

Open problem

Open problem

Recall the Fourier matrices,

$$F_M = \frac{1}{\sqrt{M}} \left[e^{2\pi ijk/M}\right].$$

We can define a tight frame by taking $N < M$ element

$$F_{N,M} = \frac{1}{\sqrt{M}} \left[e^{2\pi ijk/M}\right]_{j\in B, k=0,1,\dots,M-1}.$$

The column forms a Parseval frame with each column has length $\sqrt{N/M}$.

Kadison-Singer Theorem tells us that we can partition $\{0, 1, .., M-1\}$ into r subsets $L_1, ..., L_r$ such that

$$F(B, L) = \frac{1}{\sqrt{M}} \left[e^{2\pi ijk/M}\right]_{j\in B, k\in L_i}$$

has upper bound $(1/\sqrt{r} + \sqrt{N/M})^2$

Open problem

This is a problem related to construction of exponential frames on fractals

Consider $F_3, F_{3^2}, ..., F_{3^N}$ and consider

$$B_N = \left\{ \sum_{j=0}^{N-1} 3^j \epsilon_j : \epsilon_j \in \{0, 2\} \right\}$$

Then the Fourier submatrices $F(B_N, 3^N)$ forms a Parseval frame with column norms $2^N/3^N$.

Open problem

Question

For all $\epsilon > 0$, there exists N and L_N such that $F(B_N, L_N)$ forms a frame with frame constant

$$\frac{2^N}{3^N}(1-\epsilon), \frac{2^N}{3^N}(1+\epsilon)$$

Open problem

How about 3 is replaced by 4? Indeed,

$$B_N = \left\{ \sum_{j=0}^{N-1} 4^j \epsilon_j : \epsilon_j \in \{0,2\} \right\}, L_N = \left\{ \sum_{j=0}^{N-1} 4^j \epsilon_j : \epsilon_j \in \{0,1\} \right\}$$

Then $F(B_N, L_N)$ has a mutually orthogonal columns.
Transplanting the result to fractals, the Middle-Fourth Cantor set has exponential orthonormal basis, but there is none in Middle-Third Cantor set.

Open problem

Kadison Singer tells us that we can find a partition such that the upper bound is

$$\left(\frac{1}{\sqrt{r}}+\sqrt{\frac{2^N}{3^N}}\right)^2=\frac{2^N}{3^N}\left(1+\sqrt{\frac{3^N}{2^N r}}\right)^2.$$

However, there is no control on the lower bound. It seems that we need something stronger than KS.