

Lecture 9: Basis Pursuit and Null Space Property.

Basis Pursuit for s -sparse vector recovery is the following reconstruction scheme

$$(P_1): \min \|x\|_1 \quad \text{s.t.} \quad y = Ax.$$

(P_1) is solved rapidly by linear programming.

Our goal is to see the necessary and sufficient condition for every s -sparse vector to be solution to (P_1) .

Definition (Null space Property)

A matrix $A \in \mathbb{C}^{m \times N}$ satisfies the null space property relative to $S \subseteq [N]$ if

$$\|\vec{v}_S\|_1 < \|\vec{v}_{S^c}\|_1 \quad \forall \vec{v} \in \ker(A) \setminus \{0\}.$$

A has null space property of order s if

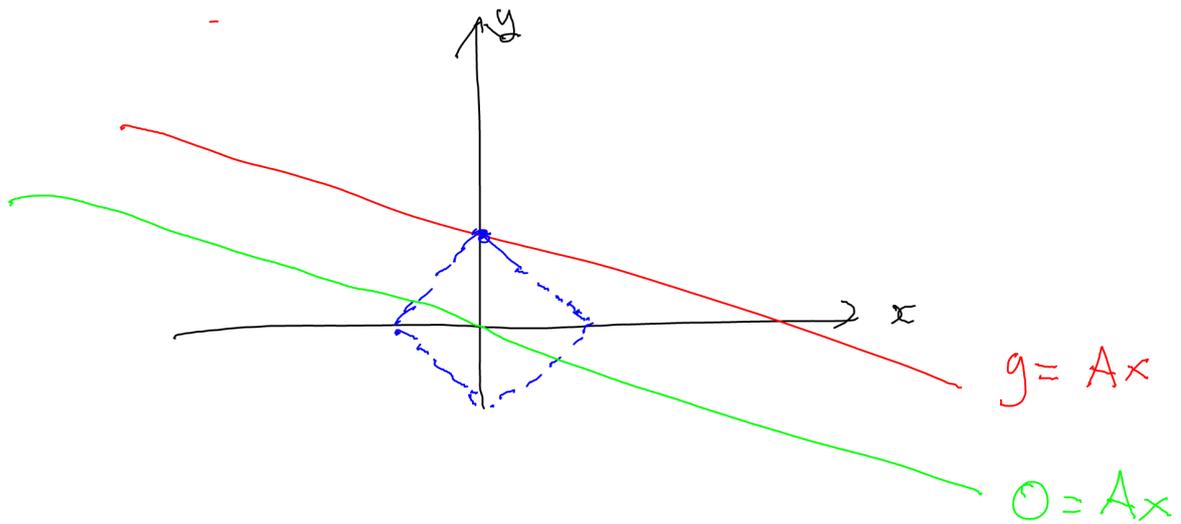
A has NSP relative to every subset $S \subseteq [N]$

and $\#S \leq s$.

Recall: $[N] = \{1, \dots, N\}$ and

$S \subset [N]$, and $\vec{v} = (v_1, \dots, v_N) \in \mathbb{C}^N$,

$$\vec{v}_S = \begin{cases} v_i & \text{if } i \in S \\ 0 & \text{if } i \in S^c. \end{cases}$$



$\ker(A)$ has NSP relative to y -axis

Theorem: Let $S \subseteq [N]$. Every $\vec{v} \in \mathbb{C}^N$ with $\text{supp } \vec{v} \subseteq S$ is the unique solution to (P_\perp) if and only if A satisfies the NSP relative to S .

Consequently, Every s -sparse vector is the unique solution to (P_\perp) if and only if A has NSP of order S .

Proof. (\Rightarrow) . If $\vec{v} \in \ker A \setminus \{0\}$ then $\vec{v} = \vec{v}_S + \vec{v}_{S^c}$
 $0 = A\vec{v} = A(\vec{v}_S) - A(-\vec{v}_{S^c})$

$$\Rightarrow A(\vec{v}_S) = A(-\vec{v}_S^c)$$

But \vec{v}_S is the unique solution to P_1 .

$$\|\vec{v}_S\|_1 < \|-\vec{v}_S^c\|_1 = \|\vec{v}_S^c\|_1.$$

(\Leftarrow) Let x be an s -sparse vector on S .

Suppose that $Ax = Az$.

Then $x - z \in \ker(A) \setminus \{0\}$.

$$\begin{aligned} \|x\|_1 &= \|x_S\|_1 \\ &\leq \|(x-z)_S\|_1 + \|z_S\|_1 \\ &< \|(x-z)_{S^c}\|_1 + \|z_S\|_1 \\ &= \|-\vec{z}_{S^c}\|_1 + \|z_S\|_1 \\ &= \|z\|_1. \quad \square \end{aligned}$$

Remark: ① Once we have NSP of order s , every s -sparse vector is the unique solution of (P_2) .

Indeed if $Ax = Az$, x, z s -sparse, unequal

Repeat the argument we did, we have

$$\|x\|_1 < \|z\|_1 \quad \text{and} \quad \|z\|_1 < \|x\|_1$$

This is a contradiction. Hence, $x = z$.

② NSP is stable under

$$\hat{A} = GA$$

where G is an invertible matrix.

$$\text{and } \tilde{A} = \begin{bmatrix} A \\ B \end{bmatrix}$$

(i.e. Introducing more measurements),

$$\text{since } \ker \hat{A} = \ker A.$$

$$\text{and } \ker \tilde{A} \subset \ker A.$$

Stable NSP

We say that a matrix $A \in \mathbb{C}^{m \times N}$ satisfies the

stable NSP of order s if $\exists 0 < \rho < 1$.

$$\|\vec{v}_s\|_2 \leq \rho \|\vec{v}_{s^c}\|_2 \quad \forall \vec{v} \in \ker A \setminus \{0\}.$$

By continuity argument on the unit ball,

it is easy to show that

NSP \Leftrightarrow Stable NSP for some ρ .

Given $x \in \mathbb{C}^N$ (may not be s -sparse), we define

x_S = vectors of x with s -largest entries remaining.

Theorem 1: Suppose that $A \in \mathbb{C}^{m \times N}$ satisfies stable NSP of order s . Then for any $\vec{x} \in \mathbb{C}^N$, the solution $x^\#$ to (P_1) : $\min \|x\|_1$ s.t. $y = Ax$ satisfies

$$\|x - x^\#\|_1 \leq \frac{2(1+p)}{1-p} \|x - x_S\|_1.$$

Remark: If x is s -sparse, $x - x_S = 0$, which recovers our previous NSP theorem.

Theorem 1 follows from the following theorem.

Theorem 2: Let $A \in \mathbb{C}^{m \times N}$. Then A satisfies stable NSP relative to S if and only if

$$\|z - x\|_1 \leq \frac{1+p}{1-p} \left(\|z\|_1 - \|x\|_1 + 2\|x_S\|_1 \right)$$

$$\forall x, z \text{ s.t. } Ax = Az.$$

Theorem 2 \Rightarrow Theorem 1, by taking $z = x^\#$ and $S = s$ largest absolute values entries of x .

$$\begin{aligned} \text{Then } \|x^\# - x\|_1 &\leq \frac{1+p}{1-p} \left(\|x^\#\|_1 - \|x\|_1 + 2\|x - x_S\|_1 \right) \\ &\leq \frac{2(1+p)}{1-p} \|x - x_S\|_1. \end{aligned}$$

We now prove Theorem 2.

(\Leftarrow) Let $v \in \ker A - \{0\}$. Let $z = 0$, $x = v$

then $Ax = Az = 0$,

$$\|v\|_1 \leq \frac{k\rho}{1-\rho} (-\|v\| + 2\|v_{sc}\|_1)$$

$$(1-\rho)\|v\|_1 \leq -(k\rho)\|v\|_1 + 2(k\rho)\|v_{sc}\|_1.$$

$$2\|v\|_1 \leq 2(k\rho)\|v_{sc}\|_1.$$

$$\|v_s\| + \|v_{sc}\|_1 \leq (k\rho)\|v_{sc}\|_1$$

$$\|v_s\|_1 \leq \rho\|v_{sc}\|_1.$$

(\Rightarrow) We need an elementary inequality.

$$\|(x-z)_{sc}\|_1 \leq \|z\|_1 - \|x\|_1 + \|(x-z)_s\|_1 + 2\|x_{sc}\|_1.$$

which follows from,

$$\|x\|_1 = \|x_s\|_1 + \|x_{sc}\|_1 \leq \|(x-z)_s\|_1 + \|z_s\|_1 + \|x_{sc}\|_1$$

$$\|(x-z)_{sc}\|_1 \leq \|x_{sc}\|_1 + \|z_{sc}\|_1$$

$$\Rightarrow \|(x-z)_{sc}\|_1 + \|x\|_1 \leq \|(x-z)_s\|_1 + 2\|x_{sc}\|_1 + \|z\|_1$$

Now $\vec{v} = x - z$

$$\text{From } \|V_S\|_1 \leq \rho \|V_{Sc}\|_1$$

$$\begin{aligned} \|V_{Sc}\|_1 &\leq \|z\|_1 - \|x\|_1 + \|(x-z)_S\|_1 + 2\|x_{Sc}\|_1 \\ &\leq \|z\|_1 - \|x\|_1 + \rho \|V_{Sc}\|_1 + 2\|x_{Sc}\|_1 \end{aligned}$$

$$\Rightarrow \|V_{Sc}\|_1 \leq \frac{1}{1-\rho} (\|z\|_1 - \|x\|_1 + 2\|x_{Sc}\|_1)$$

$$\text{Thus } \|x-z\|_1 = \|V_S\|_1 + \|V_{Sc}\|_1$$

$$\leq (1+\rho) \|V_{Sc}\|_1$$

$$\leq \frac{1+\rho}{1-\rho} (\|z\|_1 - \|x\|_1 + 2\|x_{Sc}\|_1)$$

Robustness

Sometimes measurement will have an error, we are doing minimization over a region.

$$(P_{1,\eta}) \quad \min \|x\|_1 \quad \text{st.} \quad \|y - Ax\|_2 \leq \eta.$$

We want the solution of $(P_{1,\eta}), x^\#$, to be close to our unknown vector x . in certain

norm.

Definition: We say that A satisfies the l^q -robust NSP ($1 \leq q$) of order s if there exists $0 < \rho < 1$ and $\tau > 0$ such that

$$\|\vec{v}_S\|_q \leq \frac{\rho}{s^{1-\frac{1}{q}}} \|\vec{v}_{S^c}\|_q + \tau \|A\vec{v}\|_2 \quad \forall \vec{v} \in \mathbb{R}^N,$$

for any $S \subset [N]$ with $\#S \leq s$.

Remark: l^2 robust \Rightarrow stable NSP.

Remark: If $\eta = 0$, x is s -sparse, we have the original result, that (P_{\perp}) recovers the s -sparse solution.

Now, we first consider l^1 -robust NSP.

Proposition: Let $A \in \mathbb{C}^{m \times N}$. Then A satisfies the l^1 -Robust NSP of order s if and only if

$$\|z - x\|_1 \leq \frac{1+\rho}{1-\rho} \left(\|z\|_1 - \|x\|_1 + 2\|x - x_S\|_1 \right) + \frac{2\tau}{1-\rho} \|A(z-x)\|.$$

Proof: (\Leftarrow) Let $z = V_S c$ $x = -V_S$

$$\text{Then } \|v\|_1 \leq \frac{kp}{1-p} (\|V_S c\|_1 - \|V_S\|_1) + \frac{2\tau}{1-p} \|Av\|$$

Rearranging the terms.

$$\|V_S\|_1 \leq p \|V_S c\|_1 + \tau \|Av\|,$$

(\Rightarrow) Let $v = z - x$,

$$\|V_S\|_1 \leq p \|V_S c\|_1 + \tau \|Av\|$$

and

$$\|V_S c\|_1 \leq \|z\|_1 - \|x\|_1 + \|V_S\|_1 + 2\|x_S c\|_1$$

$$\Rightarrow \|V_S c\|_1 \leq \frac{1}{1-p} (\|z\|_1 - \|x\|_1 + 2\|x_S c\|_1 + \tau \|Av\|)$$

$$\text{Hence } \|v\|_1 \leq \|V_S\|_1 + \|V_S c\|_1$$

$$\leq (1+p) \|V_S c\|_1$$

$$\leq \frac{kp}{1-p} (\|z\|_1 - \|x\|_1 + 2\|x_S c\|_1)$$

$$+ \frac{2\tau}{1-p} \|Av\|.$$

□

A more natural error metric should be the $\|\cdot\|_2$. We note that

Now, l^2 Robust $\Rightarrow l^1$ Robust

$$\begin{aligned} \|\vec{V}_S\|_1 &\leq S^{\frac{1}{2}} \|V_S\|_2 \\ &\leq \rho \|V_{S^c}\|_1 + \tau S^{\frac{1}{2}} \|Av\|_2. \end{aligned}$$

Lemma: Let $x \in \mathbb{C}^N$ and $S =$ index of largest s entries of x . Then for any $q > p > 0$,

$$\|x - x_S\|_q \leq \frac{1}{S^{\frac{1}{p} - \frac{1}{q}}} \|x\|_p.$$

P.f. Rearranging the coordinates of x in non-increasing order $x_1^* \geq \dots \geq x_N^*$ and $x_i^* = x_{\pi(i)}$ for some permutation $\pi(i)$.

$$\begin{aligned} \|x - x_S\|_q^q &= \sum_{i=S+1}^N |x_i^*|^q \\ &\leq |x_S^*|^{q-p} \sum_{i=S+1}^N |x_i^*|^p \\ &= \left(\frac{1}{S} \sum_{i \in S} |x_i^*|^p \right)^{\frac{q-p}{p}} \sum_{i=S+1}^N |x_i^*|^p \end{aligned}$$

$$\leq \left(\frac{1}{s} \sum_{i=1}^s |x_i^\#|^p \right)^{\frac{q-p}{p}} \sum_{i=s+1}^N |x_i^\#|^p$$

$$\leq \frac{1}{s^{\frac{q-p}{p}}} \|x\|_p^{q-p} \cdot \|x\|_p^p$$

$$= \frac{1}{s^{\frac{q-p}{p}}} \|x\|_p^q$$

$$\Rightarrow \|x - x_s\|_q \leq \frac{1}{s^{\frac{1}{p} - \frac{1}{q}}} \|x\|_p.$$

Theorem 3: Suppose that $A \in \mathbb{C}^{m \times N}$ satisfies the

l_2 -robust NSP of order s with constant μ, τ

Then for any x , the solution $x^\#$ of (P_{1,γ}).

approximates x as

$$\|x - x^\#\|_p \leq \frac{C}{s^{\frac{1}{p}}} \|x - x_s\|_1 + D s^{\frac{1}{p} - \frac{1}{2}} \gamma, \quad 1 \leq p \leq 2$$

In particular,

$$(p=1) \quad \|x - x^\#\|_1 \leq C \|x - x_s\|_1 + D \sqrt{s} \gamma$$

$$(p=2) \quad \|x - x^\#\|_2 \leq \frac{C}{\sqrt{s}} \|x - x_s\|_1 + D \gamma.$$

Proof: We show that

$$\|z-x\|_p \leq \frac{C}{S^{1-\frac{1}{p}}} \left(\|z\|_1 - \|x\|_1 + 2 \|x-x_S\|_1 \right) \\ + D S^{\frac{1}{p}-\frac{1}{2}} \|A(z-x)\|$$

Then Theorem 3 follows.

$$\|z-x\|_p \leq \|(z-x)_S\|_p + \|(z-x)_{S^c}\|_p$$

By choosing S to be the largest s entries of $z-x$,

$$\|z-x\|_p \leq \|(z-x)_S\|_p + \frac{1}{S^{1-\frac{1}{p}}} \|z-x\|_1.$$

As $z-x \in \ker A - \{0\}$

And l^2 -robust NSP $\Rightarrow l^p$ -robust NSP.

for $1 \leq p \leq 2$.

$$\|v_S\|_p^p \leq \left(\sum_{i \in S} |v_i|^p \right) \\ \leq \left(\sum_{i \in S} |v_i|^2 \right)^{\frac{p}{2}} \cdot \left(\sum_{i \in S} 1^q \right)^{\frac{1}{q}}. \quad \frac{1}{\frac{2}{p}} + \frac{1}{q} = 1$$

$$\frac{p}{2} + \frac{1}{q} = 1$$

$$\frac{1}{q} = 1 - \frac{p}{2} = \frac{2-p}{2}$$

$$q = \frac{2}{2-p}$$

$$\|V_S\|_p^p \leq \|V_S\|_2^p \cdot S^{\frac{2-p}{2}}$$

$$\|V_S\|_p \leq \|V_S\|_2 \cdot S^{(1-\frac{p}{2}) \cdot \frac{1}{p}}$$

$$= \|V_S\|_2 S^{\frac{1}{p} - \frac{1}{2}}$$

$$\leq S^{\frac{1}{p} - \frac{1}{2}} \left(\frac{p \|V_S\|_1}{S^{1-\frac{1}{2}}} + \tau \|Av\| \right)$$

$$= \frac{p}{S^{1-\frac{1}{p}}} \|V_S\|_1 + \tau S^{\frac{1}{p} - \frac{1}{2}} \|Av\|$$

Hence,

$$\|z-x\|_p \leq \|(z-x)_S\|_p + \|(z-x)\|_1 \cdot \frac{1}{S^{1-\frac{1}{p}}}$$

$$\leq \frac{p}{S^{1-\frac{1}{p}}} \|(z-x)_S\|_1 + \tau S^{\frac{1}{p} - \frac{1}{2}} \|A(z-x)\| + \|(z-x)\|_1 \cdot \frac{1}{S^{1-\frac{1}{p}}}$$

$$\leq \frac{(1+p)}{S^{1-\frac{1}{p}}} \|z-x\|_1 + \tau S^{\frac{1}{p} - \frac{1}{2}} \|A(z-x)\|$$

By l^1 - Robust NSP

$$\begin{aligned} &\leq \frac{4\rho}{S^{1-\frac{1}{p}}} \left(\frac{4\rho}{1-\rho} \left(\|z\|_1 - \|x\|_1 + 2\|x_{S^c}\|_1 \right) \right. \\ &\quad \left. + \frac{2\tau S^{1-\frac{1}{2}}}{1-\rho} \|A(z-x)\| \right) \\ &\quad + \tau S^{\frac{1}{p}-\frac{1}{2}} \|A(z-x)\| \\ &= \frac{(4\rho)^2}{1-\rho} \cdot \frac{1}{S^{1-\frac{1}{p}}} \left(\|z\|_1 - \|x\|_1 + 2\|x_{S^c}\|_1 \right) \\ &\quad + \frac{(3+\rho)\tau}{1-\rho} S^{\frac{1}{p}-\frac{1}{2}} \|A(z-x)\|. \end{aligned}$$

This establishes the whole proof. \square